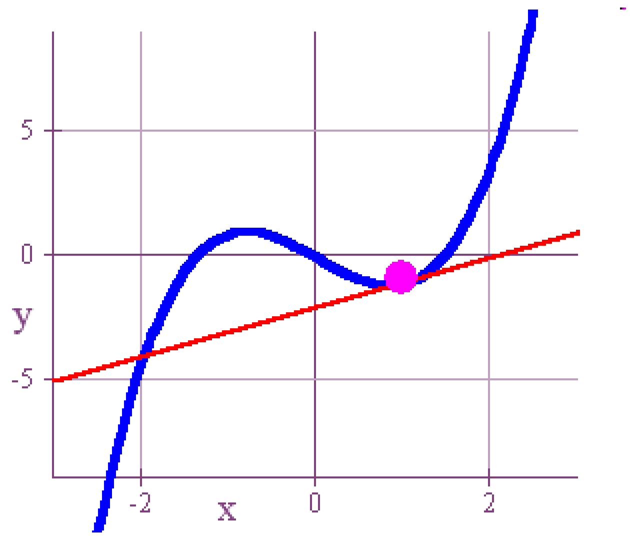
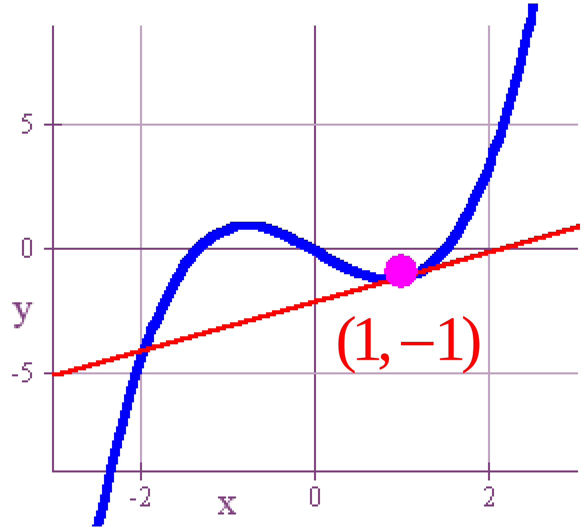
TANGENT LINES

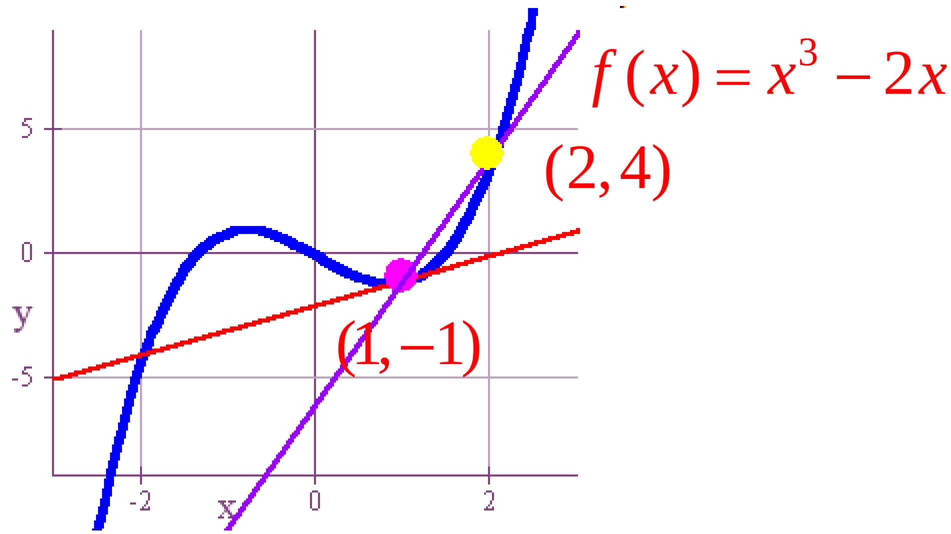


A few hundred years ago, one of the important questions in mathematics was, "How do you define a tangent line at an arbitrary point on a curve?"

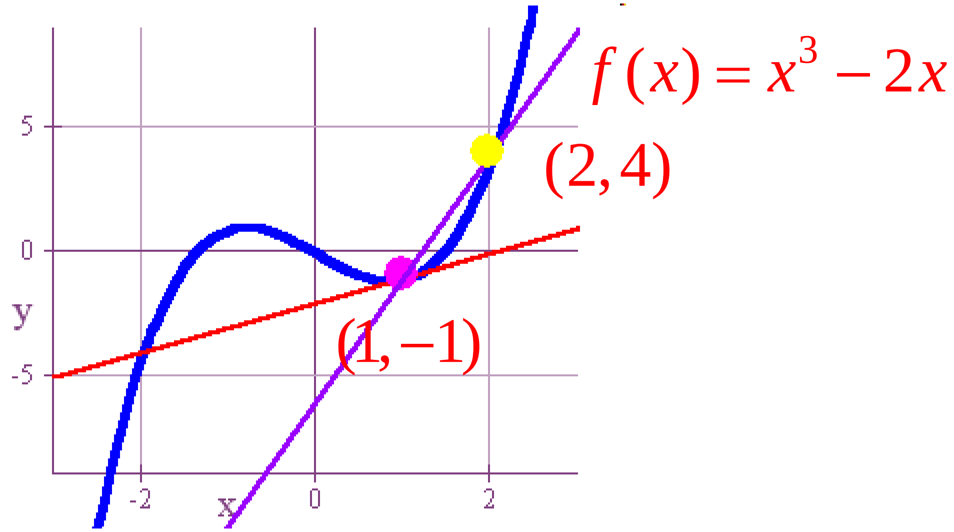


$f(x) = x^3 - 2x$

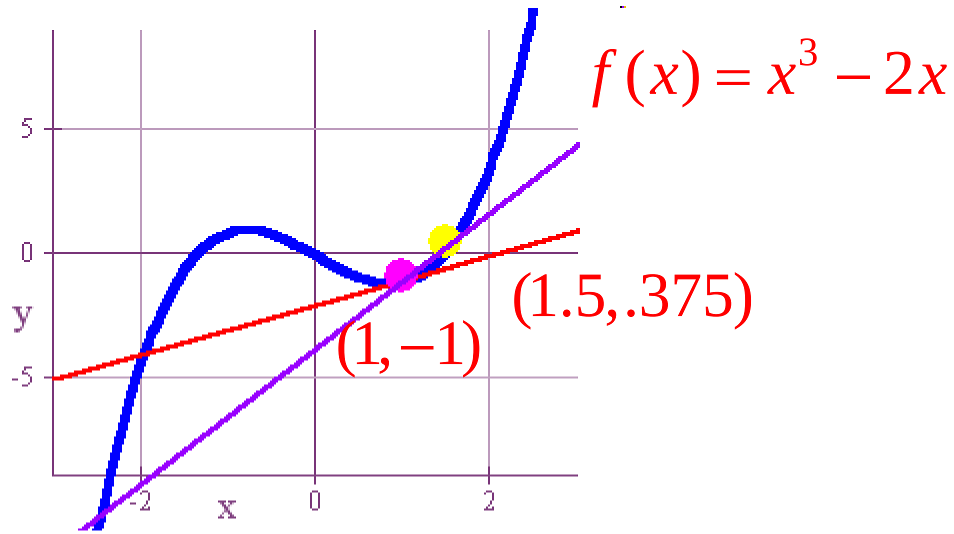
One way to go about it is through a process of successive approximations. For example, we can approximate the tangent line below by taking a second point close to (1,-1) and constructing the secant line through (1,-1) and our second point.



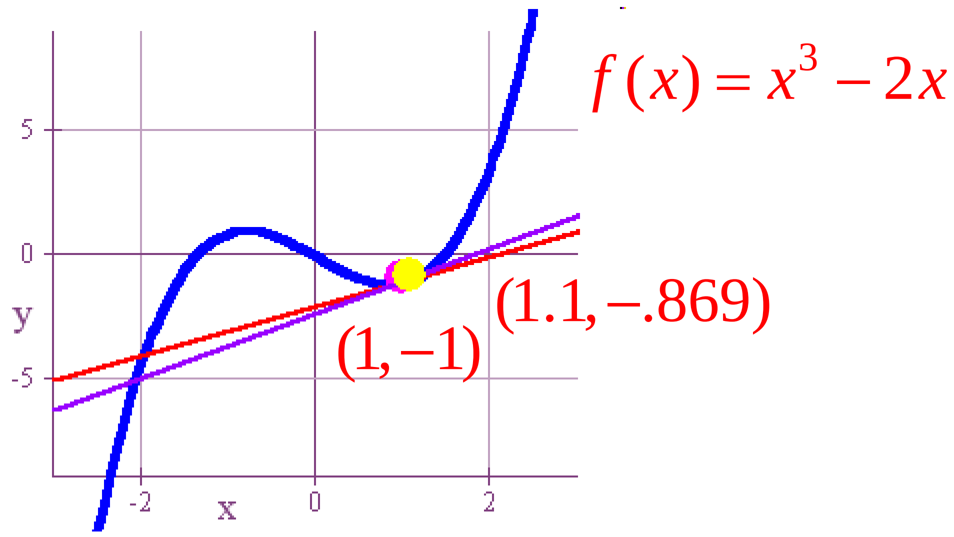
Well, clearly our second line doesn't approximate the tangent line all that well, but if we move our second point even closer to (1,-1), we'll get a better approximation.



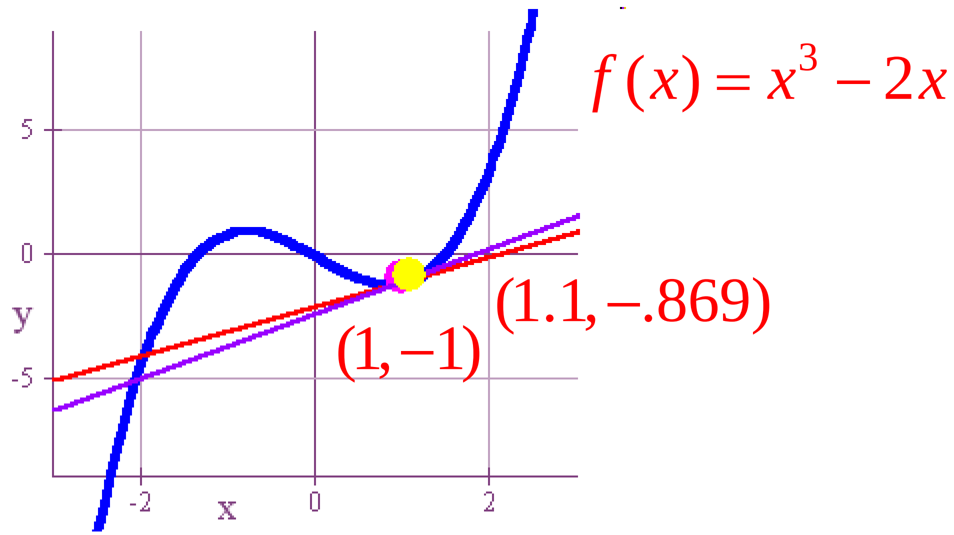
And here is the result. If we look closely, we can see that the line through (1,-1) an (1.5,.375) is a much closer approximation of the tangent line than what our first try gave us.



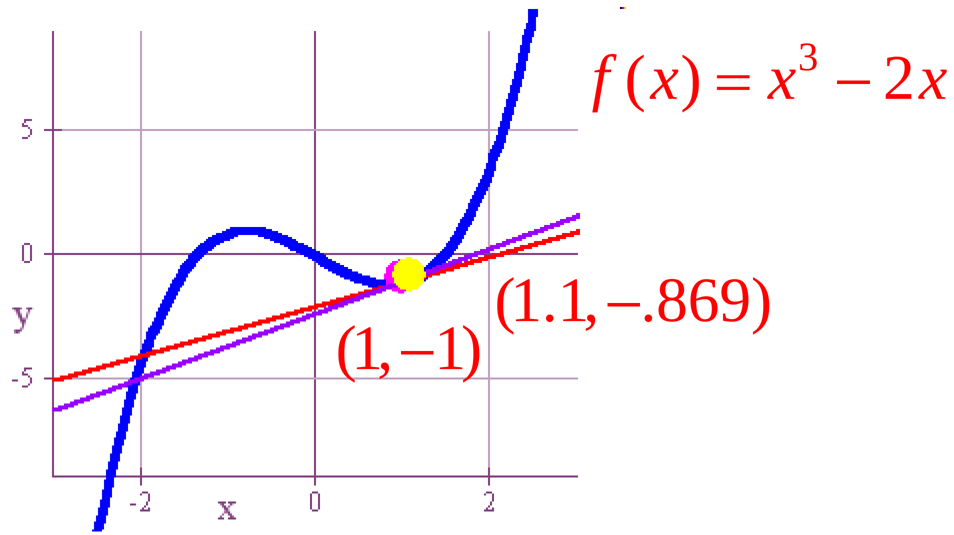
If we now look at the line through (1,-1) an (1.1,-.869), we won't find much difference between this and the actual tangent line.



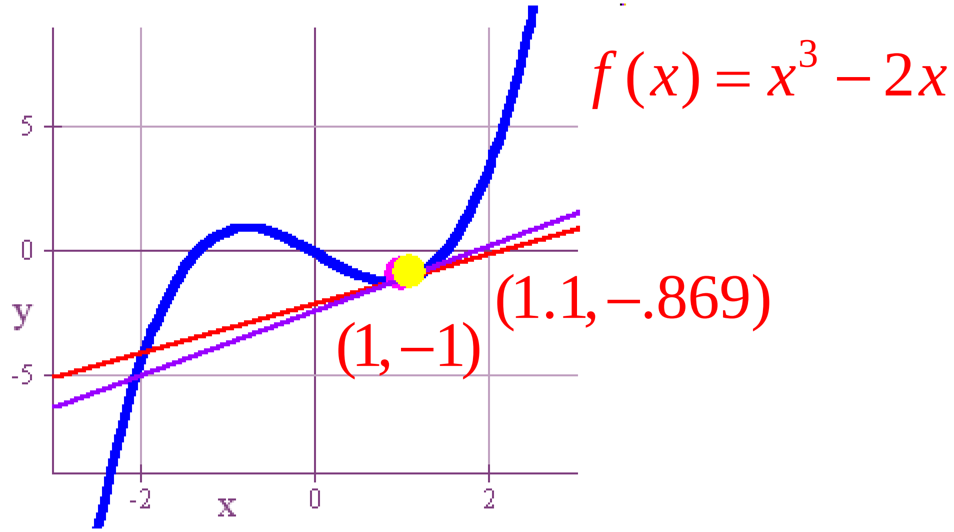
So this is a process which works. If we fix one point on the curve at (1,-1), and move a second point closer and closer to (1,-1), then the line through these two points will get closer to the actual tangent line.



Now let's refine our process.



We know that the tangent line we're looking for passes through (1,-1), so if we can figure out its exact slope, then we're in business!



Well, if our second point has coordinates (x,f(x)), then we can use our slope formula to get the slope of the line through this point and (1,-1).

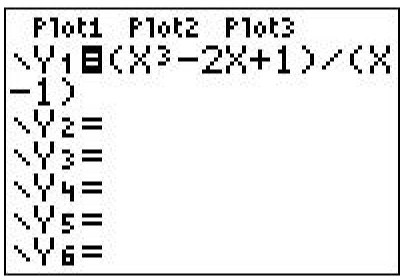
$$f(x) = x^3 - 2x$$

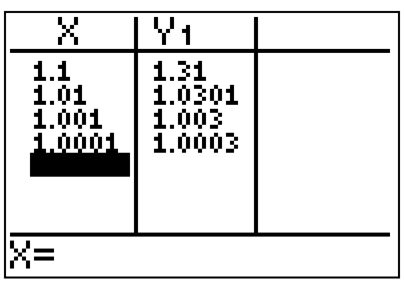
$$\text{slope} = \frac{f(x) - f(1)}{x - 1} = \frac{x^3 - 2x + 1}{x - 1}$$

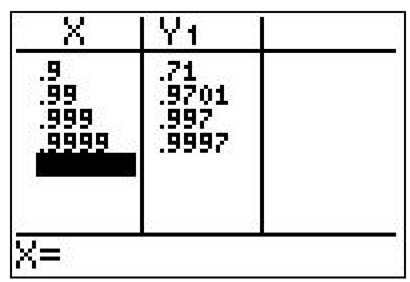
Now the question is what happens to the slope as we let *x* get closer and closer to 1? Fortunately, we can use our calculator to help us figure this out.

$f(x) = x^3 - 2x$

$$\text{slope} = \frac{f(x) - f(1)}{x - 1} = \frac{x^3 - 2x + 1}{x - 1}$$



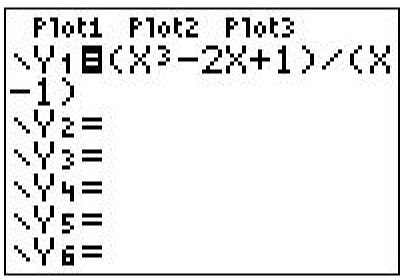


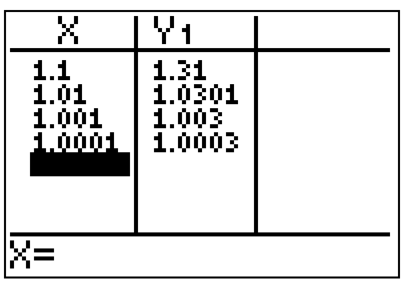


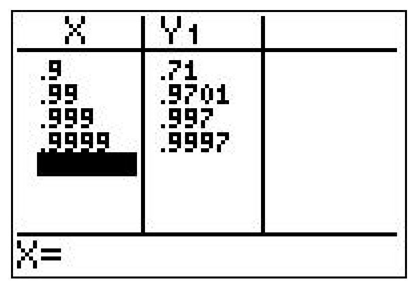
It looks like our approximation is getting closer and closer to 1. Thus, we'll assume that the tangent line that passes through (1,-1) has slope 1.

$$f(x) = x^3 - 2x$$

$$\text{slope} = \frac{f(x) - f(1)}{x - 1} = \frac{x^3 - 2x + 1}{x - 1}$$







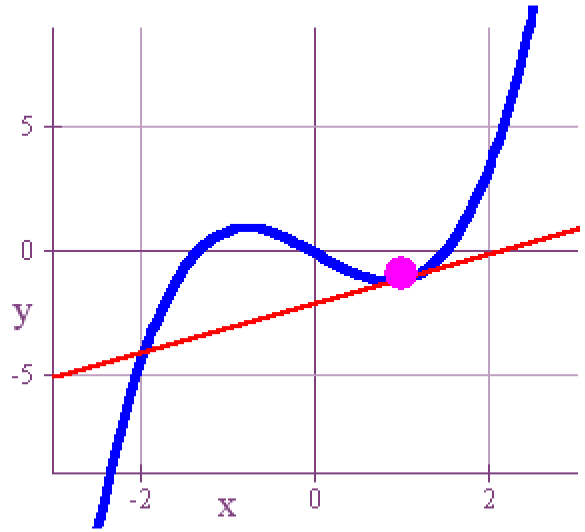
And it works!

$$f(x) = x^3 - 2x$$

$$P = (1, -1)$$

$$\text{slope } = m = 1$$

$$\text{tangent } = T = 1 \cdot (x-1) - 1 = x - 2$$

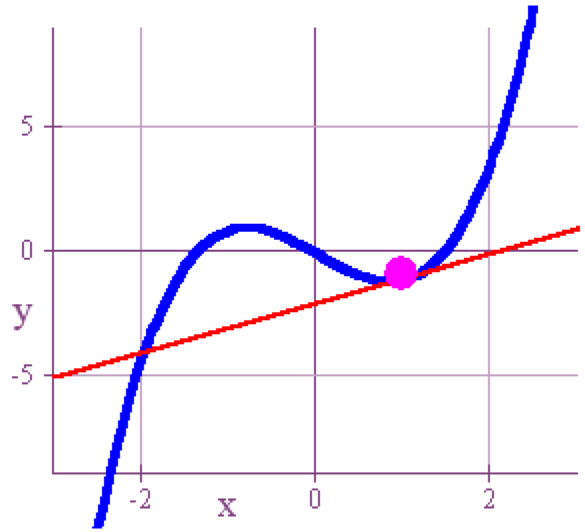


Now let's review what we did. Essentially, we undertook a series of successive approximations that we call a "limit process."

$f(x) = x^3 - 2x$

$$\text{slope of secant line } = \frac{f(x) - f(1)}{x - 1}$$

$$\text{slope of tangent line } = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

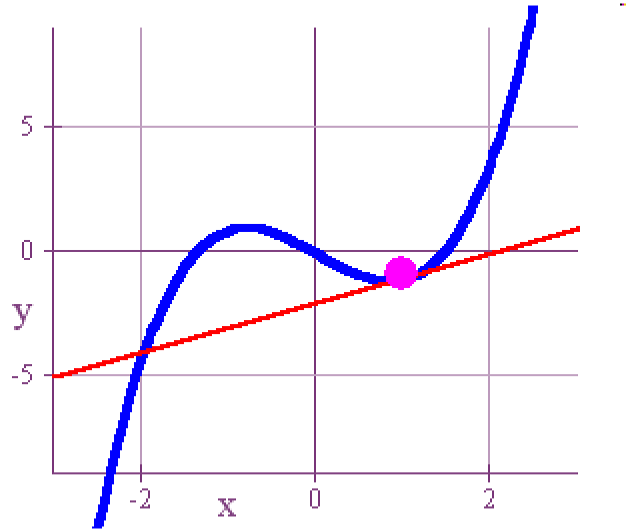


In this particular limit process, notice that we can't set *x =1* without resulting in division by zero.

$$f(x) = x^3 - 2x$$

$$\text{slope of secant line } = \frac{f(x) - f(1)}{x - 1}$$

$$\text{slope of tangent line } = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

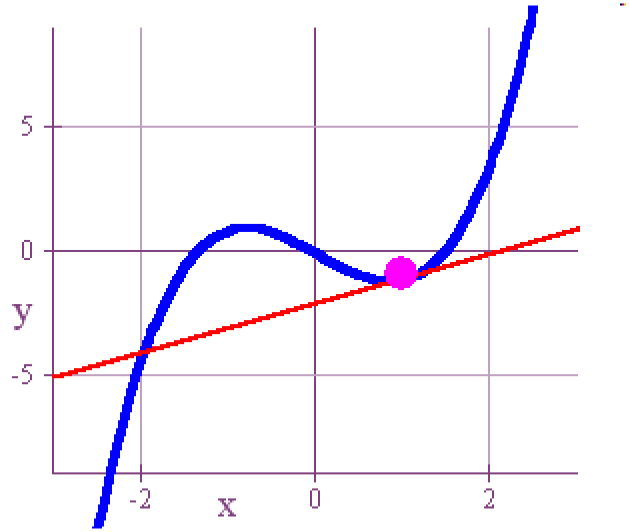


Thus, we ask ourselves what our expression gets close to as *x* gets closer to, but not equal to, 1.

$$f(x) = x^3 - 2x$$

$$\text{slope of secant line } = \frac{f(x) - f(1)}{x - 1}$$

$$\text{slope of tangent line } = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

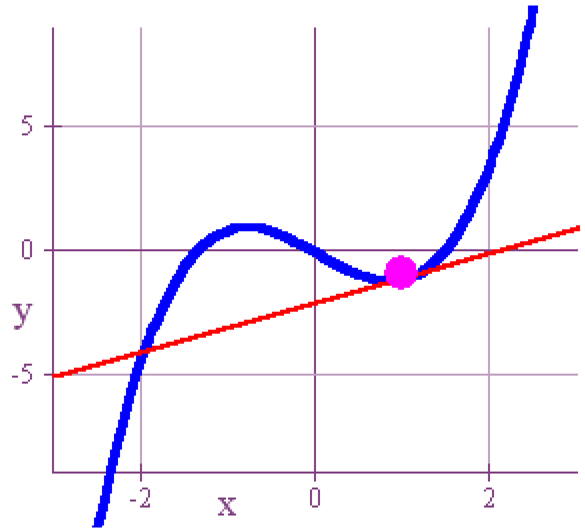


If our expression gets close to a particular value as *x* gets closer to, but not equal to, 1, then we call this value the limit of the expression a *x* approaches 1.

$f(x) = x^3 - 2x$

$$\text{slope of secant line } = \frac{f(x) - f(1)}{x - 1}$$

$$\text{slope of tangent line } = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

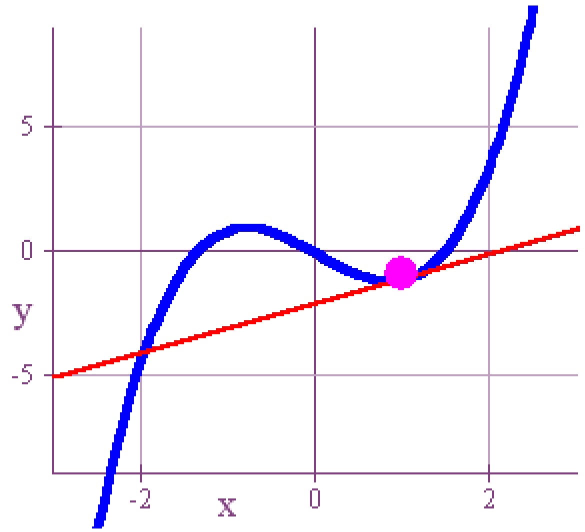


At this point you might say, "Who cares? I want practical applications!" Well, don't worry. Finding the slope of a tangent line does have practical applications!

$$f(x) = x^3 - 2x$$

$$\text{slope of secant line} = \frac{f(x) - f(1)}{x - 1}$$

$$\text{slope of tangent line} = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

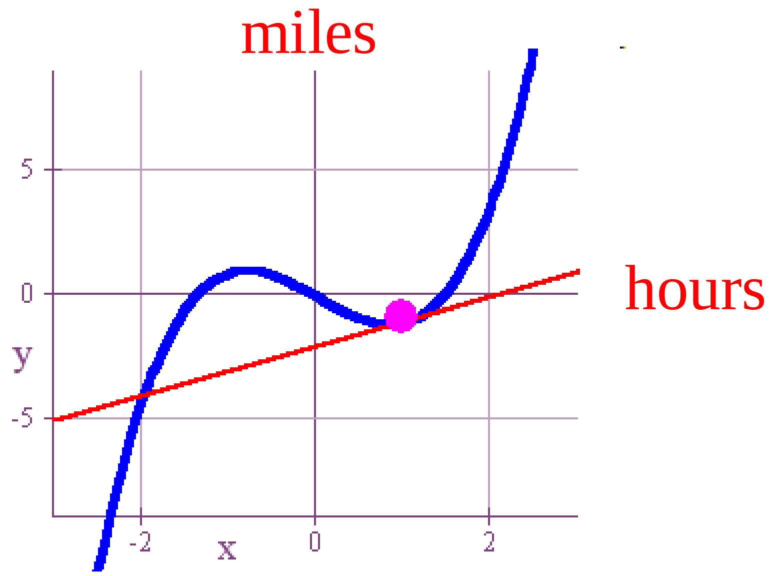


For example, suppose our input variable represents time in hours, and our output variable represents distance traveled in miles.

$f(x) = x^3 - 2x$

$$\text{slope of secant line } = \frac{f(x) - f(1)}{x - 1}$$

$$\text{slope of tangent line } = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

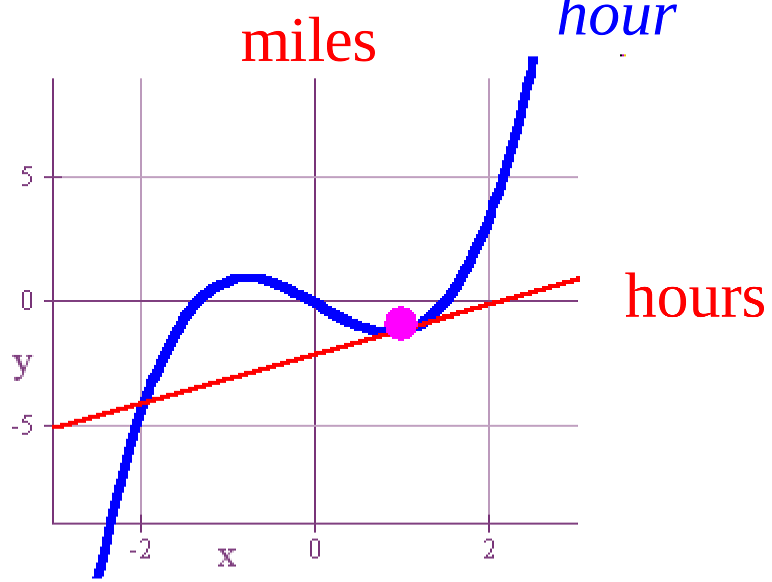


In this case, the slope of the tangent line represents our velocity in miles per hour at a particular moment in time, or what we call our *instantaneous velocity*!

$f(x) = x^3 - 2x$

$$\text{slope of tangent line} = \lim_{\text{change in time} \to 0} \frac{\text{change in distance}}{\text{change in time}}$$

$$= \text{instantaneous velocity in } \frac{miles}{hour}$$



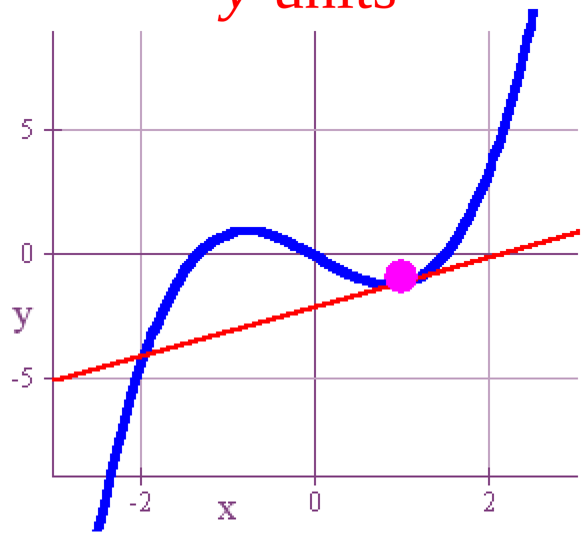
In fact, a limit process can be used to find any *instantaneous rate of change*.

$f(x) = x^3 - 2x$

$$\text{slope of tangent line} = \lim_{\text{change in } x \to 0} \frac{\text{change in } y}{\text{change in } x}$$

$$= \text{instantaneous rate of change in } \frac{y \text{ units}}{x \text{ units}}$$

y units



x units

Thus, calculus begins with a general investigation of the limit process. We'll start our next lecture with the definition below.

Let $f(x)$ be a function. Then the limit of $f(x)$ as x approaches a equals L is defined as follows:

$\lim_{x \to a} f(x) = L$ means that as x gets close to, but not equal to, a, the values of $f(x)$ get closer and closer to L.