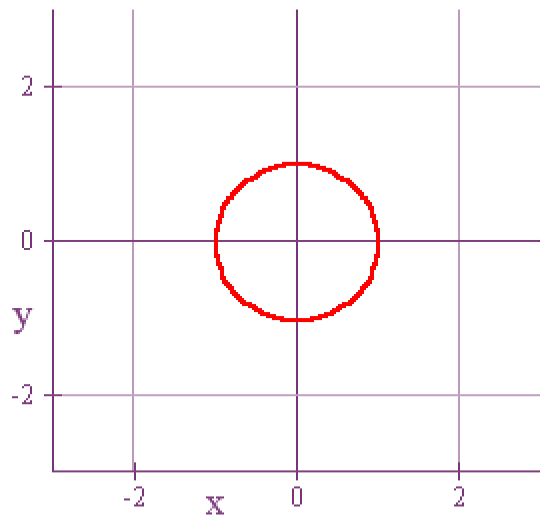
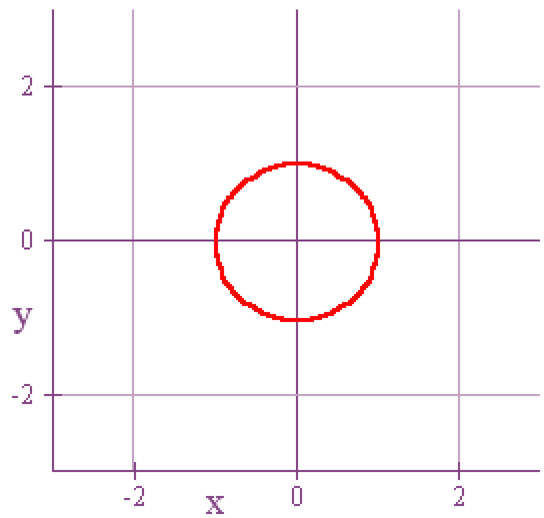
RELATED RATES

Suppose we have a circle of radius *r*. Then we all know the formula that gives us the relationship between the radius of the circle and its area.

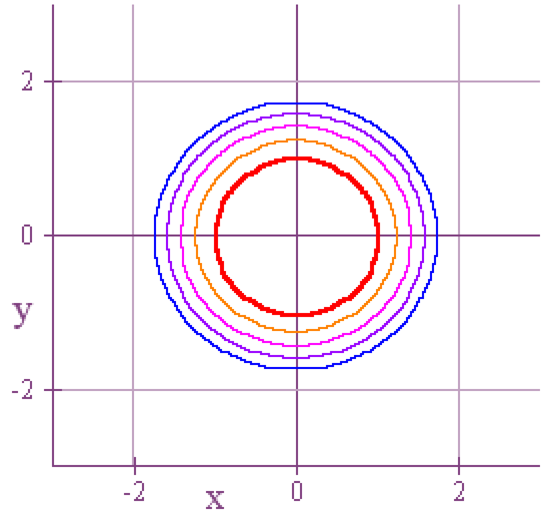


Suppose we have a circle of radius *r*. Then we all know the formula that gives us the relationship between the radius of the circle and its area.



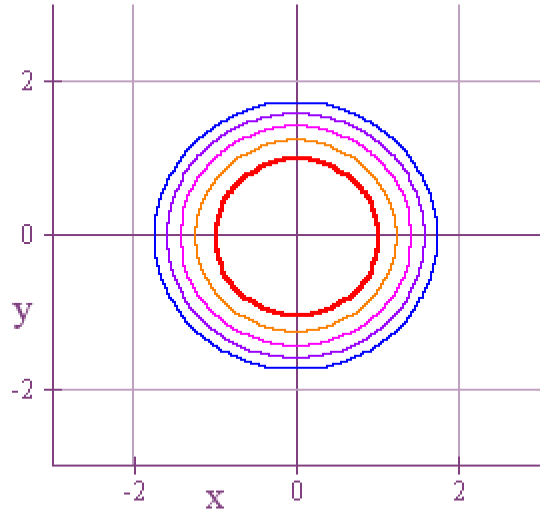
$$A = \pi r^2$$

Now suppose that you create a circle by tossing a pebble into a pond. The result will now be that you see a circular wave that gets bigger over time.



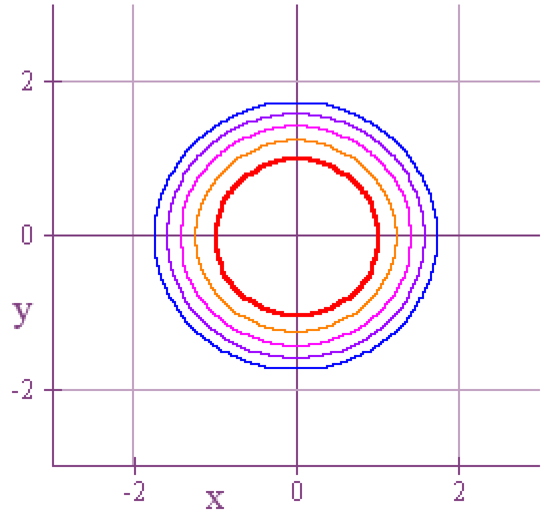
$$A = \pi r^2$$

This means that both *area* and *radius* are now changing over time. In other words, they are now both *functions* of time.



$$A(t) = \pi[r(t)]^2$$

However, just as *area* and *radius* are related by a formula, so will the rates of change of *area* and *radius* over time be related by a formula.

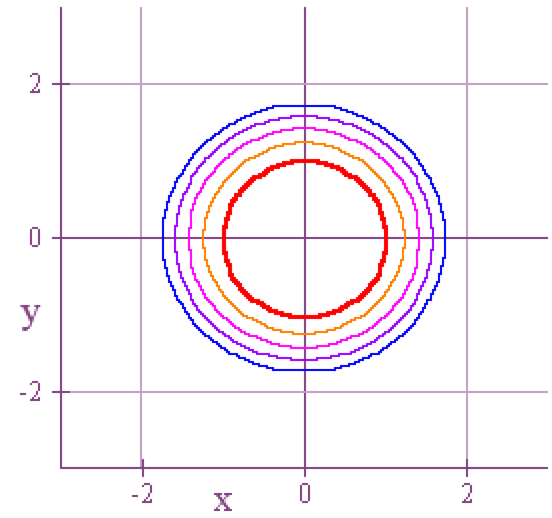


$$A(t) = \pi[r(t)]^2$$

$$\frac{dA(t)}{dt} = \frac{d\pi[r(t)]^2}{dt}$$

And this is how we work it out using the *chain rule*.

$$A(t) = \pi[r(t)]^2$$



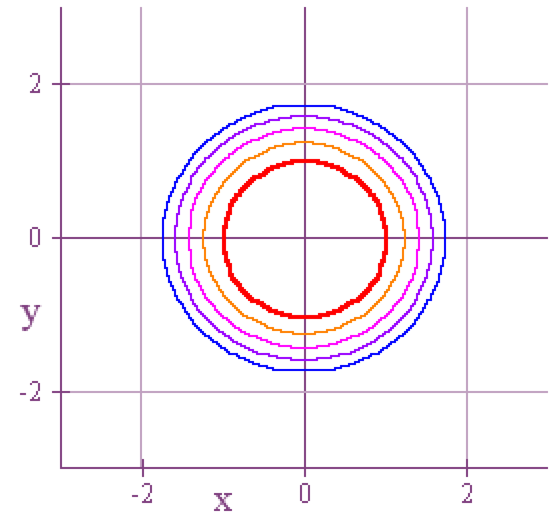
$$\Rightarrow \frac{dA(t)}{dt} = \frac{d\pi[r(t)]^2}{dt}$$

$$\Rightarrow \frac{dA(t)}{dt} = \pi \cdot 2r(t)\frac{dr(t)}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r\frac{dr}{dt}$$

Also, we usually streamline the notation as follows.

$$A = \pi r^2$$



$$\Rightarrow \frac{dA}{dt} = \frac{d\pi r^2}{dt}$$

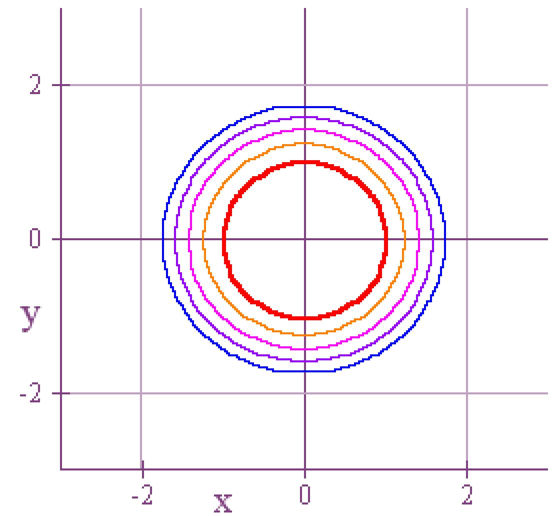
$$\Rightarrow \frac{dA}{dt} = \pi \cdot 2r \frac{dr}{dt}$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

We usually call this type of problem a *related rates* problem.

$$A = \pi r^2$$

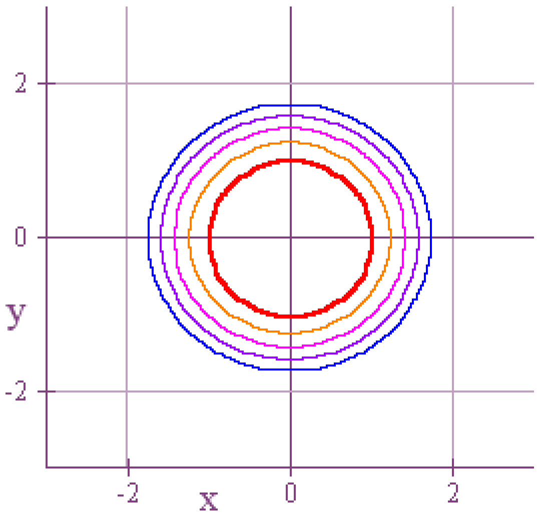
$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$



In other words, if two variables are related by a formula, and if those same variables are also, for example, changing over time, then their rates of change with respect to time will also be related.

$$A = \pi r^2$$

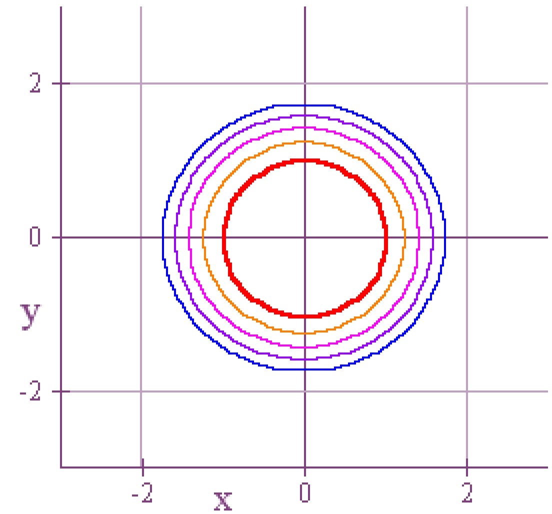
$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$



For instance, suppose that the radius of the circle is increasing at a rate of 2 feet/second. Then what is the rate at which the area is increasing when the radius is 4 feet?

$$A = \pi r^2$$

$$\Rightarrow \frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$



For instance, suppose that the radius of the circle is increasing at a rate of 2 feet/second. Then what is the rate at which the area is increasing when the radius is 4 feet?

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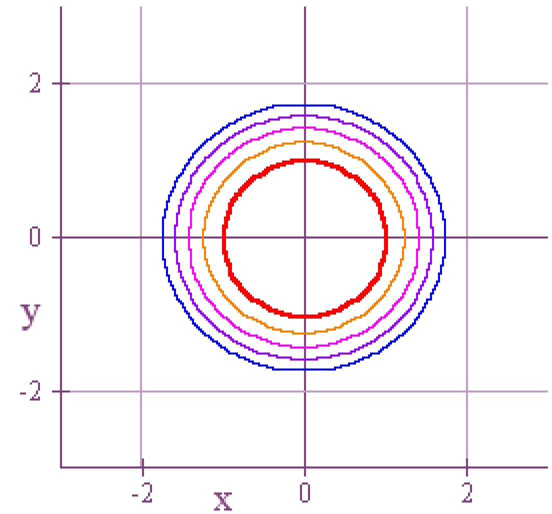
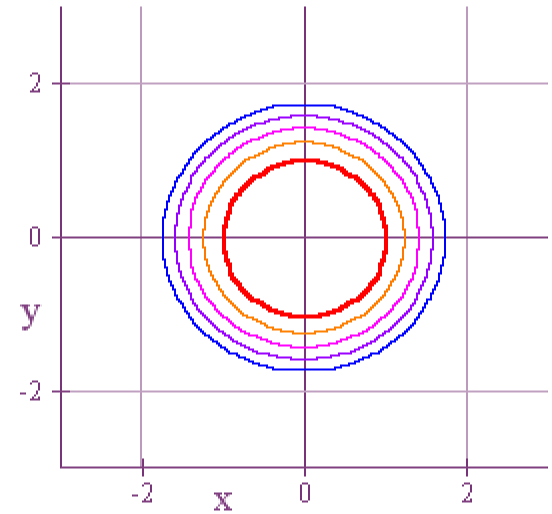


$$\frac{dA}{dt} = 2\pi(4 \text{ feet})(2 \text{ feet/second}) = 16\pi \text{ feet}^2 / \text{second}$$

$$\approx 50.27 \text{ feet}^2 / \text{second}$$

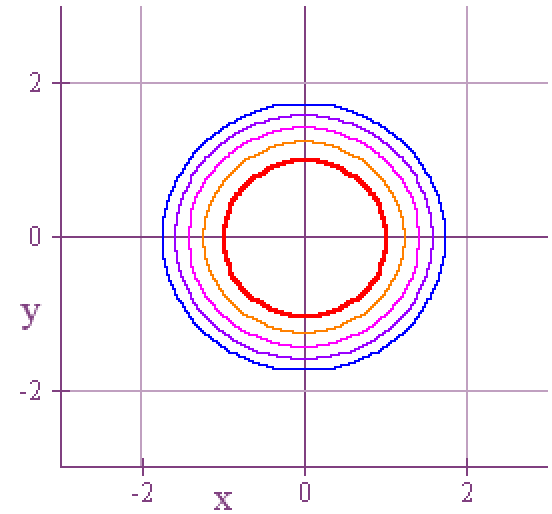
For instance, suppose that the radius of the circle is increasing at a rate of 2 feet/second. **Let's take this same scenario and figure out the rate at which the circumference of the circle is changing when *r = 4 feet.***

$$C = 2\pi r$$



For instance, suppose that the radius of the circle is increasing at a rate of 2 feet/second. Let's take this same scenario and now figure out the rate at which the circumference of the circle is changing when *r = 4 feet.*

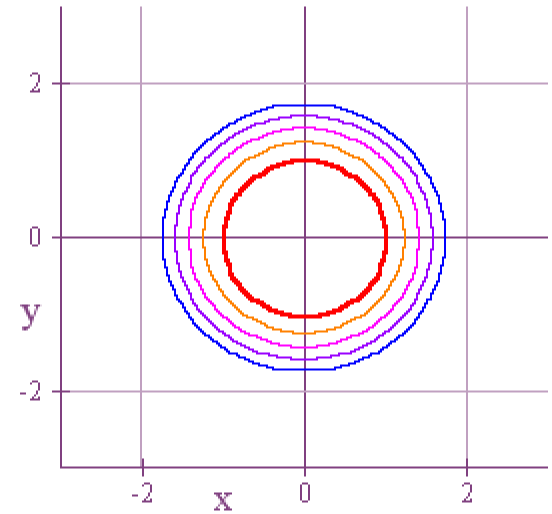
$$C = 2\pi r$$



$$\Rightarrow \frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

For instance, suppose that the radius of the circle is increasing at a rate of 2 feet/second. Let's take this same scenario and now figure out the rate at which the circumference of the circle is changing when *r* = *4 feet.*

$$C = 2\pi r$$



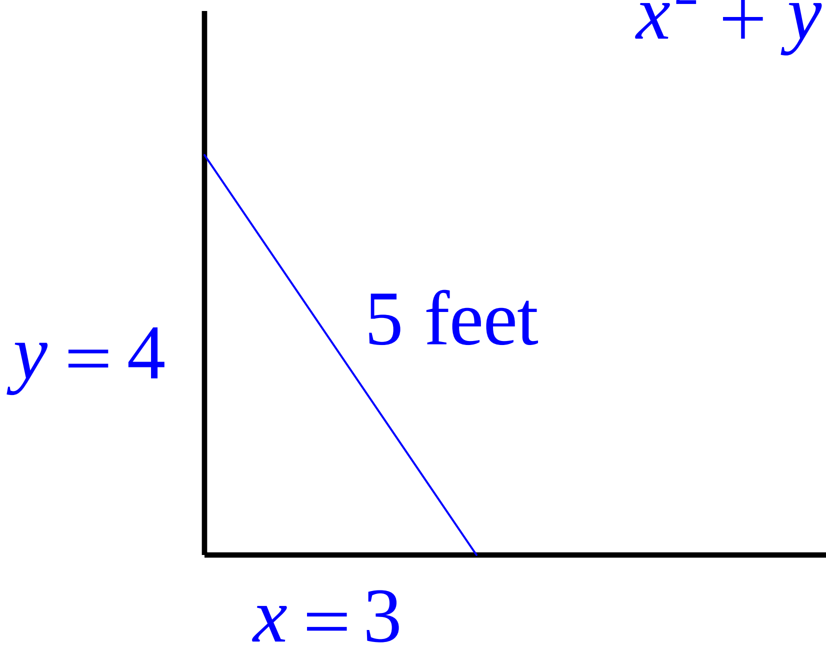
$$\Rightarrow \frac{dC}{dt} = 2\pi \frac{dr}{dt}$$

$$\frac{dC}{dt} = 2\pi(2\text{ feet/second}) = 4\pi \text{ feet / second}$$

$$\approx 12.57 \text{ feet / second}$$

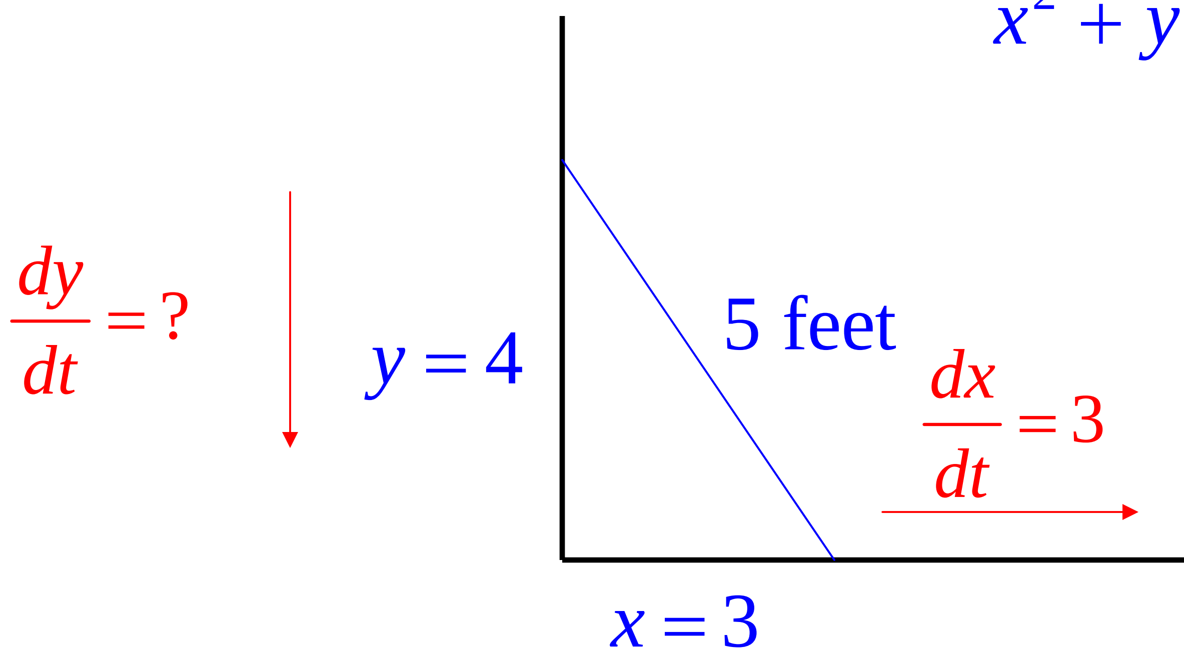
Now suppose we have a 5 foot ladder leaning against a wall. Suppose also that the base of the ladder is 3 feet from the wall. Then by the *Pythagorean Theorem*, the top of the ladder is 4 feet from the floor.

$$x^2 + y^2 = 5^2 = 25$$



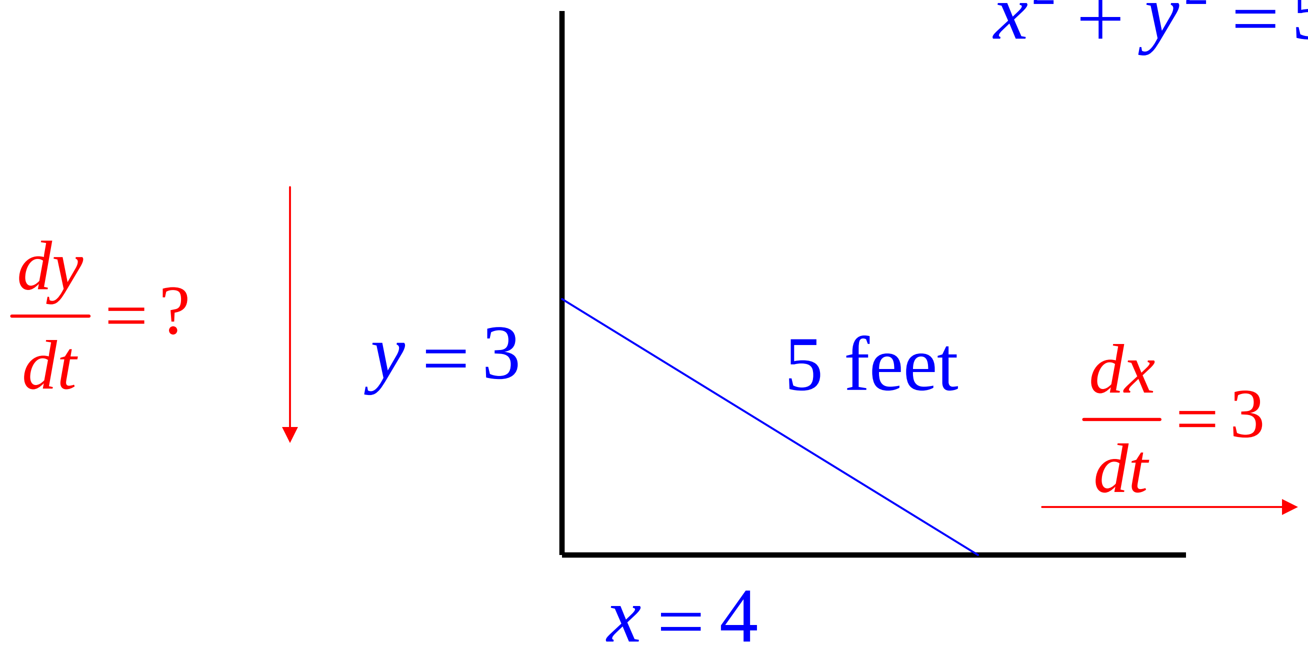
Additionally, suppose that the bottom part of the ladder is sliding away from the wall at a rate of 3 feet/second. At the moment when the bottom of the ladder is 4 feet from the wall, what is the rate at which the top of the ladder is approaching the floor?

$$x^2 + y^2 = 5^2 = 25$$



First, when *x=4*, we have by the *Pythagorean Theorem* that *y=3*.

$$x^2 + y^2 = 5^2 = 25$$



Now just differentiate both sides of the equation below with respect to *t*, plug in the values you have, and solve for *dy/dt*.

$$x = 4$$

$$x^2 + y^2 = 25$$

$$y = 3$$

$$\frac{dx}{dt} = 3$$

$$\frac{dy}{dt} = ?$$

Now just differentiate both sides of the equation below with respect to *t*, plug in the values you have, and solve for *dy/dt*.

$$x = 4$$

$$y = 3$$

$$\frac{dx}{dt} = 3$$

$$\frac{dy}{dt} = ?$$

$$x^2 + y^2 = 25$$

$$\Rightarrow \frac{d(x^2 + y^2)}{dt} = \frac{d(25)}{dt}$$

Now just differentiate both sides of the equation below with respect to *t*, plug in the values you have, and solve for *dy/dt*.

$$x = 4$$

$$y = 3$$

$$\frac{dx}{dt} = 3$$

$$\frac{dy}{dt} = ?$$

$$x^2 + y^2 = 25$$

$$\Rightarrow \frac{d(x^2 + y^2)}{dt} = \frac{d(25)}{dt}$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

Now just differentiate both sides of the equation below with respect to *t*, plug in the values you have, and solve for *dy/dt*.

$$x = 4$$

$$y = 3$$

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$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = -\frac{x}{y}\frac{dx}{dt}$$

Now just differentiate both sides of the equation below with respect to *t*, plug in the values you have, and solve for *dy/dt*.

$$x = 4$$

$$y = 3$$

$$\frac{dx}{dt} = 3$$

$$\frac{dy}{dt} = ?$$

$$x^2 + y^2 = 25$$

$$\Rightarrow \frac{d(x^2 + y^2)}{dt} = \frac{d(25)}{dt}$$

$$\Rightarrow 2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$$

$$\Rightarrow \frac{dy}{dt} = \frac{-2x\frac{dx}{dt}}{2y} = -\frac{x}{y}\frac{dx}{dt} \Rightarrow \frac{dy}{dt} = -\frac{4}{3}\cdot 3 = -4\frac{\text{feet}}{\text{second}}$$