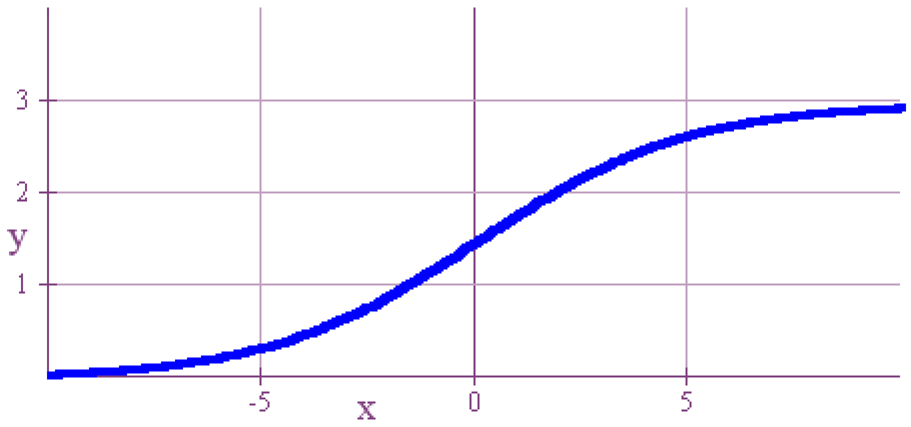
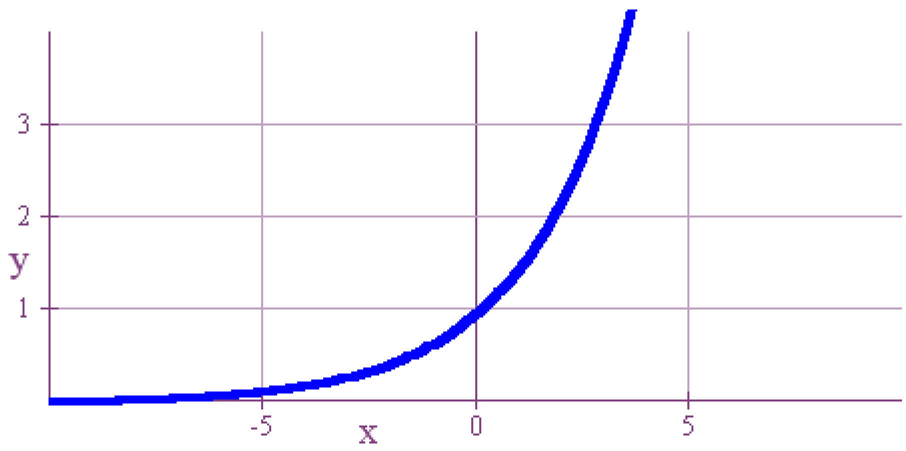
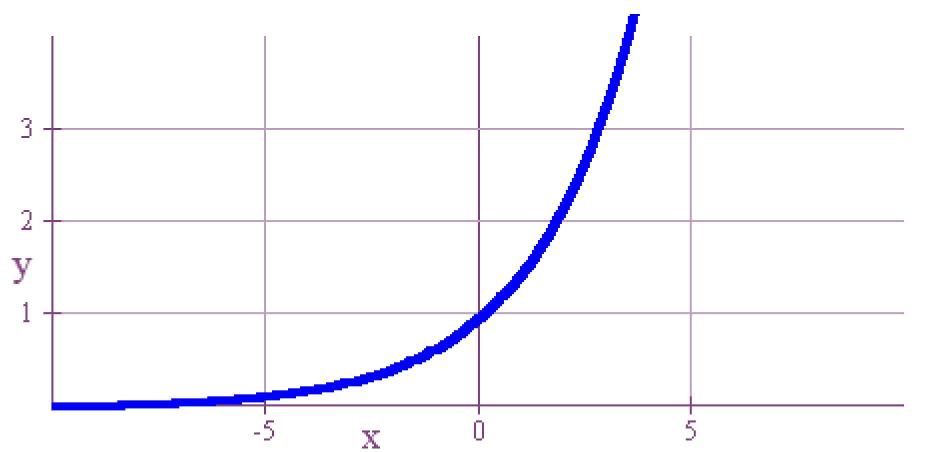
LOGISTIC FUNCTIONS



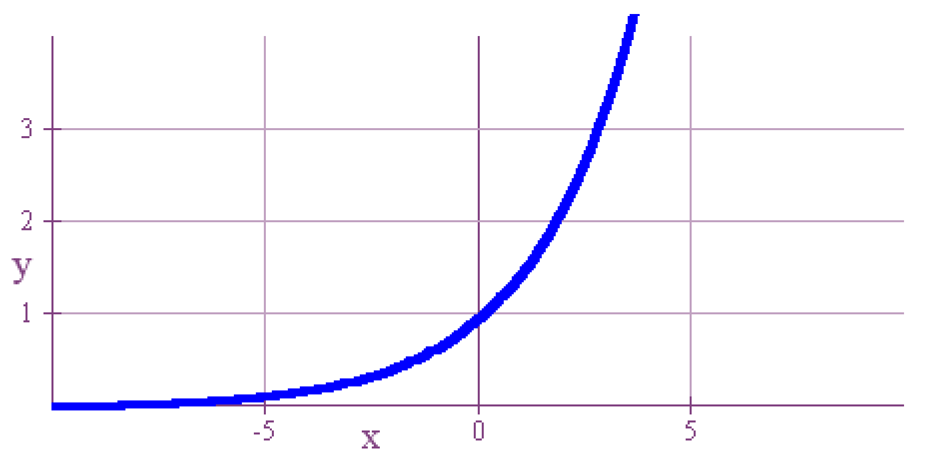
We're all familiar with the shape of the good ol' exponential growth curve.

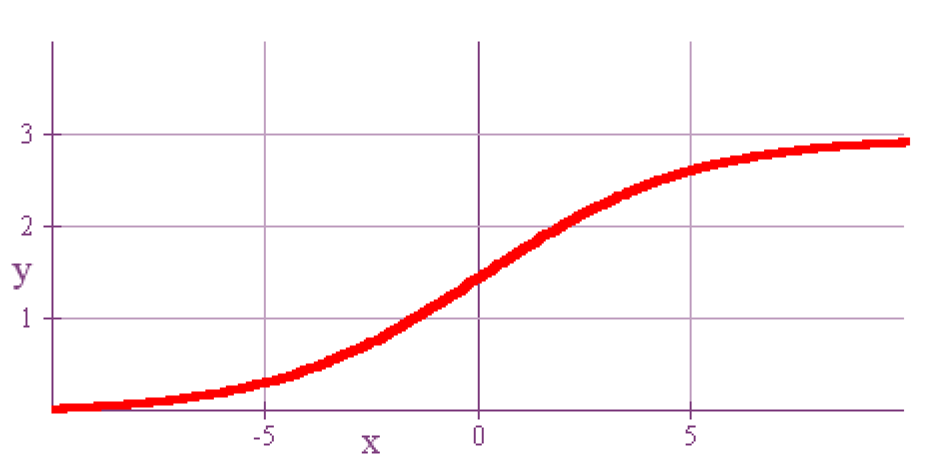


However, in the real world things aren't usually able to keep growing for ever and ever.



In the real world, there are limits to growth, and this results in a curve that we call a logistic curve.





A logistic function for a logistic curve can be written in either of two interchangeable forms.

$$y = \frac{C}{1 + A \cdot B^{-x}} \quad \text{or} \quad y = \frac{c}{1 + a \cdot e^{-bx}}$$

$A > 0$

$B > 0 \,\&\, B \neq 1$

$a > 0$

$b \neq 0$

It’s easy to convert from one form to the other.

$$y = \frac{C}{1 + A \cdot B^{-x}} \quad \text{or} \quad y = \frac{c}{1 + a \cdot e^{-bx}}$$

$A > 0$

$B > 0 \,\&\, B \neq 1$

$a > 0$

$b \neq 0$

Here we convert the first form into the second form.

$$y = \frac{2}{1+3\cdot 1.5^{-x}} = \frac{2}{1+3\cdot e^{\ln(1.5^{-x})}} = \frac{2}{1+3\cdot e^{-x\ln(1.5)}} \approx \frac{2}{1+3\cdot e^{-0.41x}}$$

$$y = \frac{C}{1+A\cdot B^{-x}} \rightarrow \qquad y = \frac{c}{1+a\cdot e^{-bx}}$$

And here we convert something in the second form back to the first form.

$$y = \frac{2}{1+3\cdot e^{-2x}} = \frac{2}{1+3\cdot\left(e^{2}\right)^{-x}} \approx \frac{2}{1+3\cdot 7.39^{-x}}$$

$$y = \frac{c}{1+a\cdot e^{-bx}} \rightarrow \qquad y = \frac{C}{1+A\cdot B^{-x}}$$

Let’s look at the following simple logistic function, and consider what happens as *x* gets large in the positive direction.

$$y = f(x) = \frac{3}{1+2^{-x}} = \frac{3}{1+\dfrac{1}{2^x}}$$

As *x* gets large, $1/2^x$ gets closer to zero, and the whole ratio gets closer to 3.

$$y = f(x) = \frac{3}{1+2^{-x}} = \frac{3}{1+\frac{1}{2^x}}$$

$$\lim_{x\to\infty} \frac{3}{1+2^{-x}} = \lim_{x\to\infty} \frac{3}{1+\frac{1}{2^x}} = 3$$

We'll call 3 the "upper limit value." In fields such as biology, this limit value is also called the "carrying capacity." Also, some just call 3 the "limit value."

$$y = f(x) = \frac{3}{1+2^{-x}} = \frac{3}{1+\frac{1}{2^x}}$$

$$\lim_{x\to\infty} \frac{3}{1+2^{-x}} = \lim_{x\to\infty} \frac{3}{1+\frac{1}{2^x}} = 3$$

Now what happens as *x* goes in the negative direction?

$$y = f(x) = \frac{3}{1+2^{-x}}$$

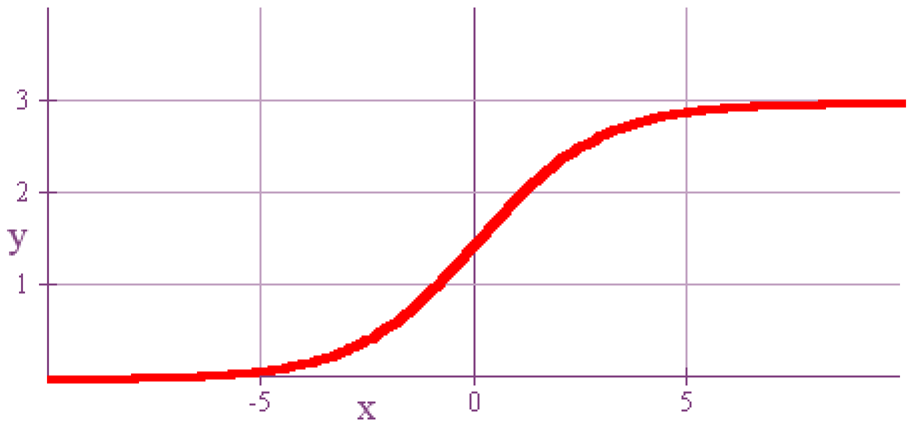
In this case, the ratio approaches zero as the denominator gets larger and larger in value. We'll call zero the "lower limit value."

$$y = f(x) = \frac{3}{1+2^{-x}}$$

$$\lim_{x \to -\infty} \frac{3}{1+2^{-x}} = 0$$

Here's the graph of our function.

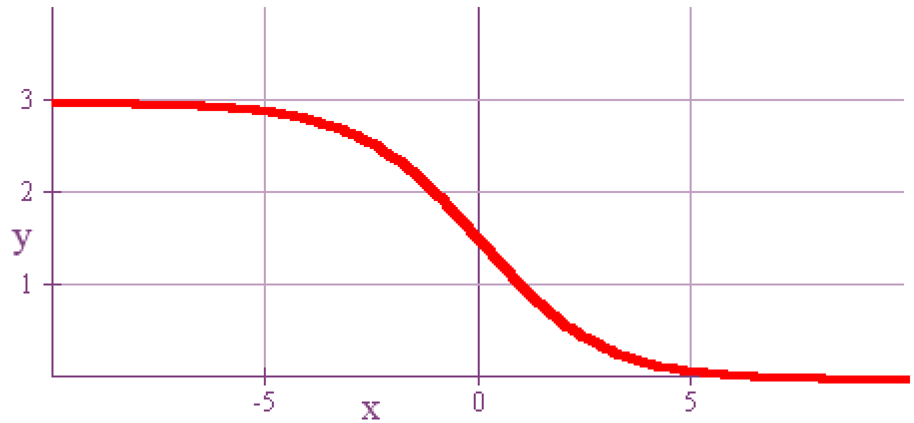
$$y = f(x) = \frac{3}{1+2^{-x}}$$



$$\lim_{x \to -\infty} \frac{3}{1+2^{-x}} = 0 \qquad \lim_{x \to \infty} \frac{3}{1+2^{-x}} = 3$$

Notice that if *B* is a fraction between 0 and 1, our graph looks a little different.

$$y = f(x) = \frac{3}{1+\left(\frac{1}{2}\right)^{-x}} = \frac{3}{1+2^{x}}$$



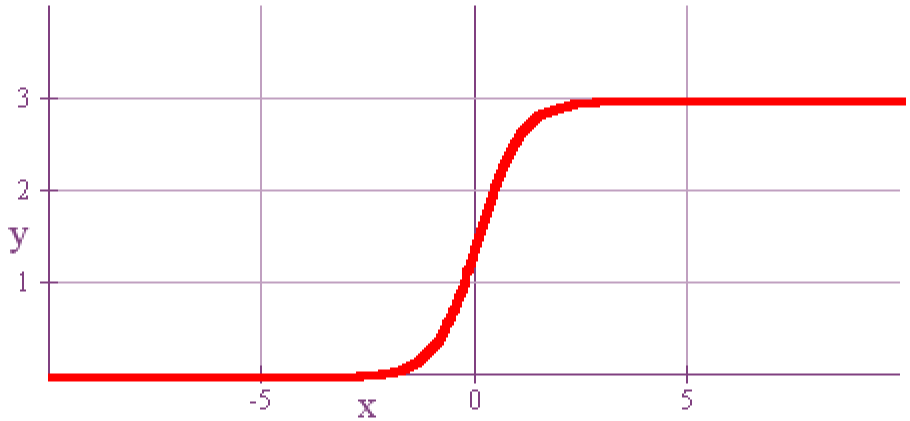
$$\lim_{x\to-\infty} \frac{3}{1+\left(\frac{1}{2}\right)^{-x}} = 3$$

$$\lim_{x\to\infty} \frac{3}{1+\left(\frac{1}{2}\right)^{-x}} = 0$$

If we use the form for our logistic function that involves *e*, then what matters is whether *b>0* or *b<0*.

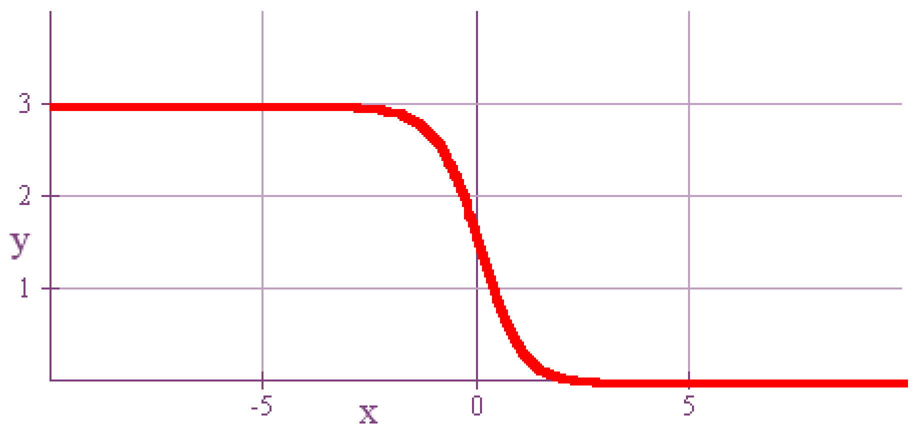
$$y = \frac{3}{1+e^{-2x}}$$

$$b = 2$$

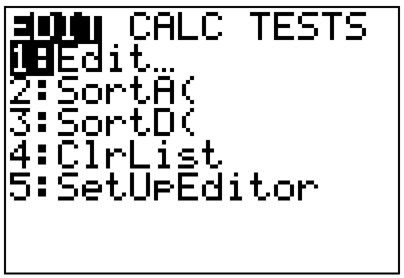


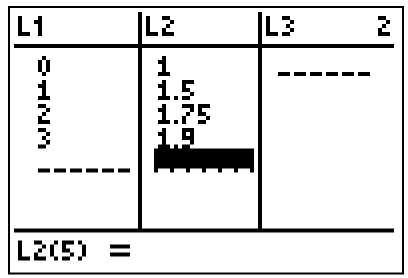
$$y = \frac{3}{1+e^{2x}}$$

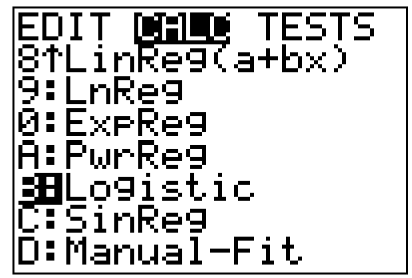
$$b = -2$$

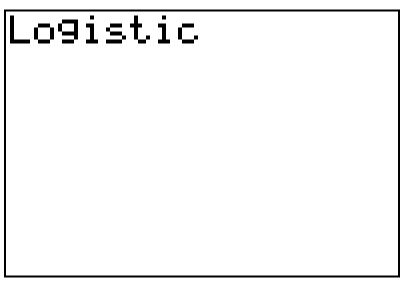


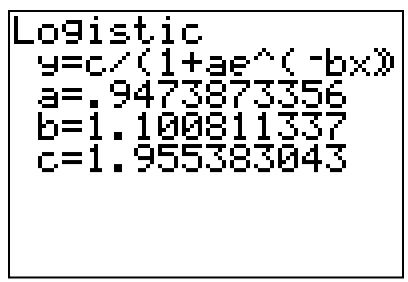
And finally, here’s how we can use our calculator to find the best fitting logistic curve for data.

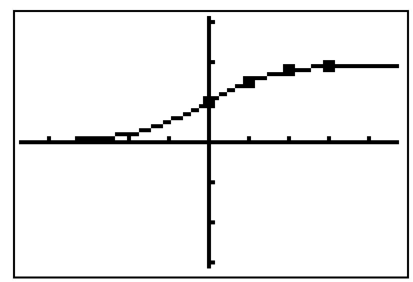












It’s all so logistical!