Finding Limits Algebraically

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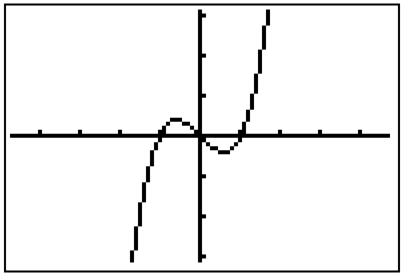
$$y = f(x) = x^3 - x$$



Graph is continuous at all points.

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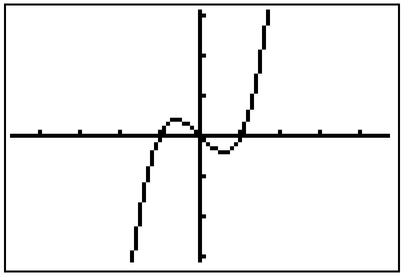

$$\lim_{x \to 2}(x^3 - x) = 2^3 - 2$$

$$= 8 - 2 = 6$$

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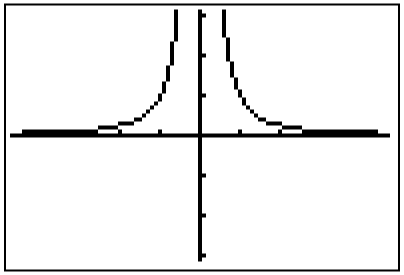
$$y = f(x) = \frac{1}{x^2}$$



Discontinuity at zero.

If our function is undefined at a point, then the graph will certainly have a discontinuity at that point Furthermore, examining the graph will often give us information about various limits.

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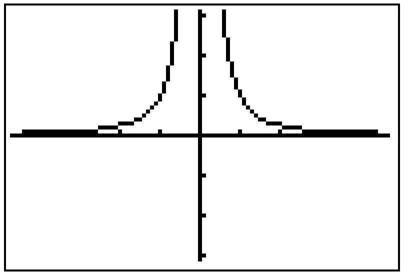
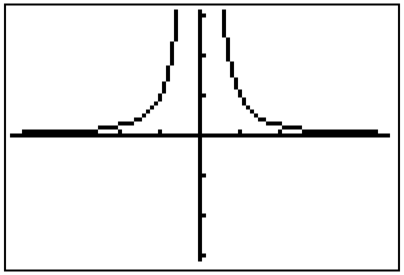


Discontinuity at zero.

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

Notice that since *infinity* is not a number, the limit actually fails to exist. However, by stating that the limit is *infinity*, we are showing specifically how it fails to exist.

$$y = f(x) = \frac{1}{x^2}$$

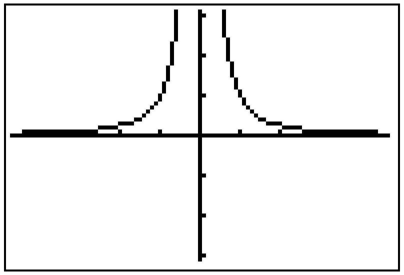


Discontinuity at zero.

$$\lim_{x \to 0} \frac{1}{x^2} = \infty$$

If we want the limit of our function as *x approaches 2*, we can just plug in 2 for *x* since the function is continuous at this point.

$$y = f(x) = \frac{1}{x^2}$$



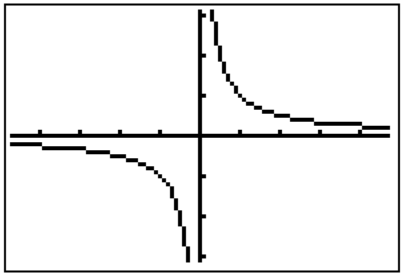
Discontinuity at zero.

Continuous at $x = 2$.

$$\lim_{x \to 2} \frac{1}{x^2} = \frac{1}{2^2} = \frac{1}{4}$$

If we look at *y = 1/x*, then we once again have a discontinuity at zero.

$$y = f(x) = \frac{1}{x}$$



Discontinuity at zero.

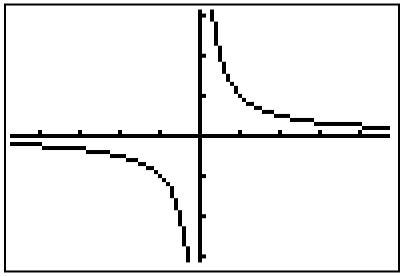
However, this time the graph goes to *+infinity* from the right hand side and *–infinity* from the left hand side. Thus, the most we can state is that the limit does not exist.

$$y = f(x) = \frac{1}{x}$$

$$\lim_{x \to 0^+} \frac{1}{x} = \infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

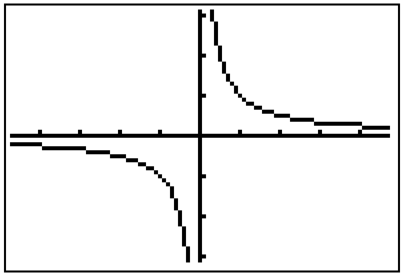
$$\lim_{x \to 0} \frac{1}{x} = \text{ does not exist}$$



Discontinuity at zero.

Nonetheless, we can still evaluate the limit at nonzero points simply by plugging in.

$$y = f(x) = \frac{1}{x}$$



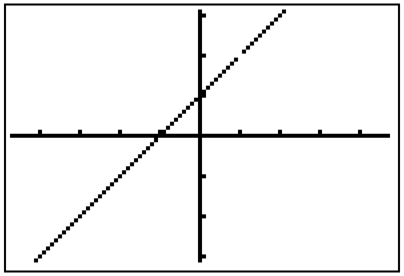
Discontinuity at zero.

$$\lim_{x \to -3} \left(\frac{1}{x} \right) = -\frac{1}{3}$$

Continuous at $x = -3$.

If a function has a discontinuity at a point, but the graph doesn't go to plus or minus infinity, then we can try simplifying the expression algebraically before evaluating. (NOTE: Graph using *Zoom Decimal*)

$$y = f(x) = \frac{x^2 - 1}{x - 1}$$



Discontinuity at 1.

If a function has a discontinuity at a point, but the graph doesn't go to plus or minus infinity, then we can try simplifying the expression algebraically before evaluating. (NOTE: Graph using *Zoom Decimal*)

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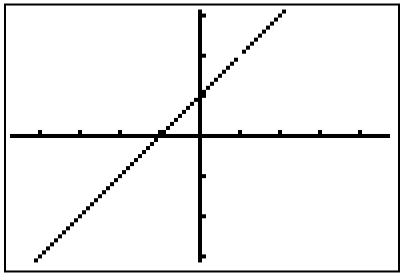
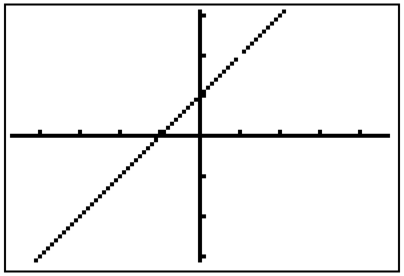


Discontinuity at 1.

$$\lim_{x \to 1} \frac{x^2 - 1}{x - 1} = \lim_{x \to 1} \frac{(x+1)(x-1)}{(x-1)} = \lim_{x \to 1}(x+1) = 1 + 1 = 2$$

Notice that this function is continuous at values that are not equal to 1, and we evaluate limits at those points by just plugging in.

$$y = f(x) = \frac{x^2 - 1}{x - 1}$$



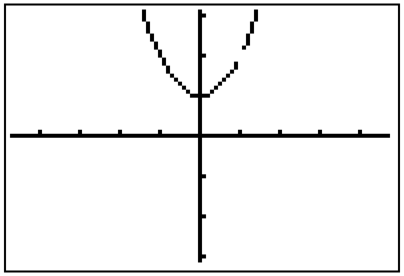
Discontinuity at 1.

Continuous at $x = 0$.

$$\lim_{x \to 0} \frac{x^2 - 1}{x - 1} = \frac{0^2 - 1}{0 - 1} = \frac{-1}{-1} = 1$$

If you are unable to simplify an expression by factoring, then try long division.

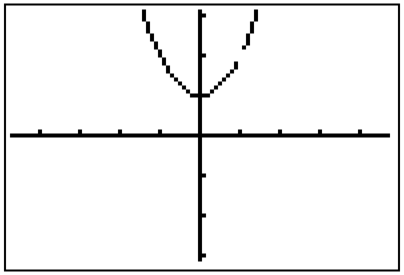
$$y = f(x) = \frac{x^3 - x^2 + x - 1}{x - 1}$$



Discontinuity at 1.

$$\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1} = ?$$

$$\begin{array}{r} x^2 + 1 \\ x - 1 \overline{) x^3 - x^2 + x - 1} \\ \underline{-x^3 + x^2 \quad\quad\quad} \\ x - 1 \\ \underline{-x + 1} \\ 0 \end{array}$$

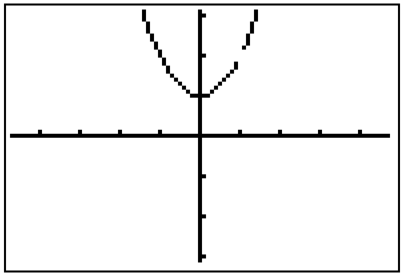


Discontinuity at 1.

$$\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1} = \lim_{x \to 1} (x^2 + 1) = 1^2 + 1 = 2$$

Of course, an alternative way to solve this problem is by using *factoring by grouping*.

$$y = f(x) = \frac{x^3 - x^2 + x - 1}{x - 1}$$

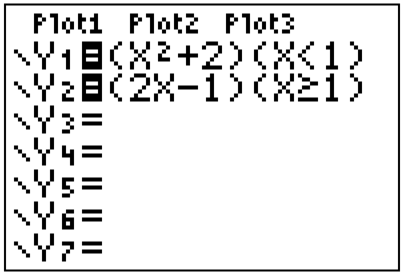


Discontinuity at 1.

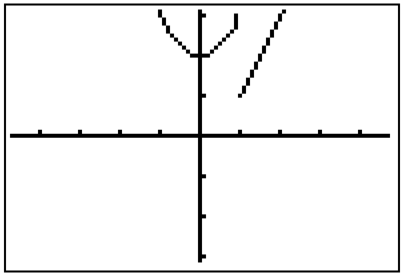
$$\lim_{x \to 1} \frac{x^3 - x^2 + x - 1}{x - 1} = \lim_{x \to 1} \frac{x^2(x-1) + 1 \cdot (x-1)}{x - 1}$$

$$= \lim_{x \to 1} \frac{(x^2+1)(x-1)}{(x-1)} = \lim_{x \to 1}(x^2+1) = 1^2 + 1 = 2$$

Another type of problem you might have to deal with is a *piecewise-defined function.*

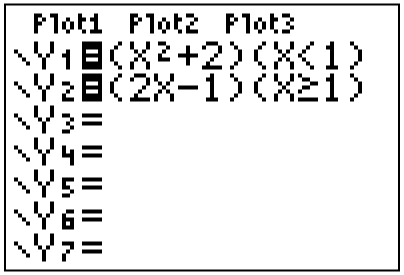


$$y = f(x) = \begin{cases} x^2 + 2 & \text{if } x < 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$$



Discontinuity at 1.

Here it's easy to see that the general limit fails to exist as ***x approaches 1.***



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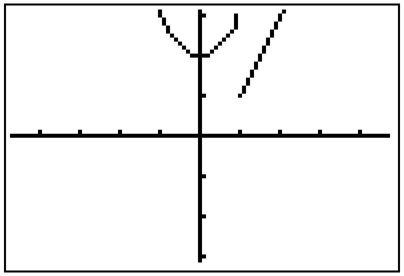
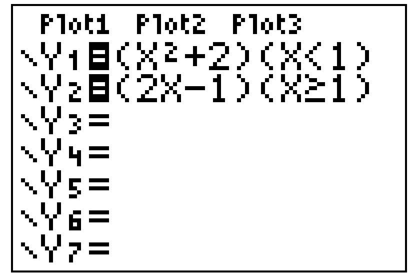


Discontinuity at 1.

We can also look at the two one-sided limits separately and show that they lead to different values.



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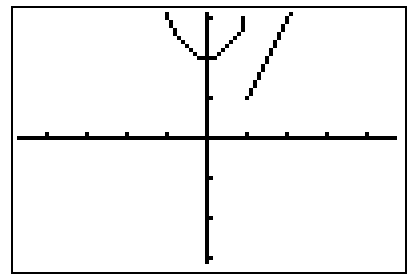

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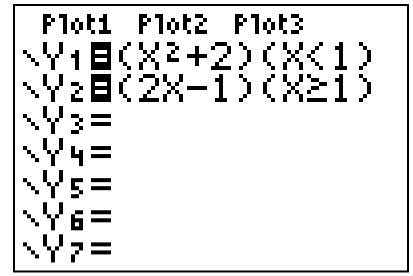
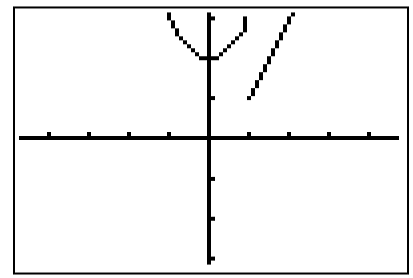


$$y = f(x) = \begin{cases} x^2 + 2 & \text{if } x < 1 \\ 2x - 1 & \text{if } x \geq 1 \end{cases}$$



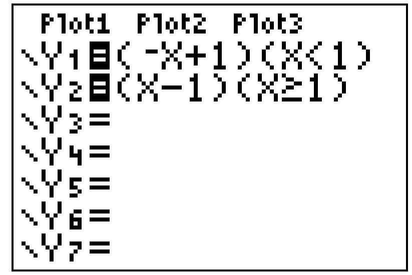
Discontinuity at 1.

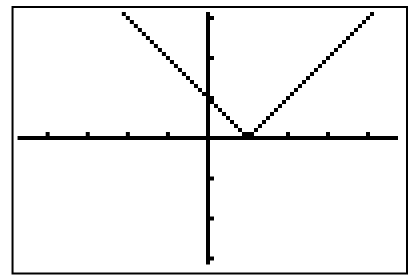
$$\lim_{x \to 1^-} (x^2 + 2) = 1^2 + 2 = 3$$

$$\lim_{x \to 1^+} (2x - 1) = 2 \cdot 1 - 1 = 1$$

$$\lim_{x \to 1} f(x) = \text{ does not exist}$$

On the other hand, if our pieces are connected, then the function is continuous and you can just plug in values to get the limit.





$$y = f(x) = \begin{cases} -x+1 & \text{if } x<1 \\ x-1 & \text{if } x \geq 1 \end{cases}$$

$$\lim_{x\to 1^-}(-x+1) = -1+1 = 0$$

$$\lim_{x\to 1^+}(x-1) = 1-1 = 0$$

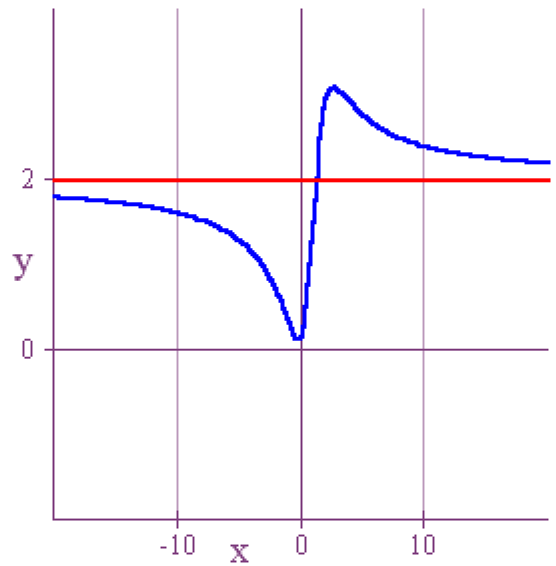
$$\lim_{x\to 1} f(x) = 0$$

If you are taking a limit of a rational function as *x* goes to plus or minus infinity, then recall that the long term behavior of the function is determined by the ratio of the leading terms.

$$\lim_{x\to\infty}\frac{4x^2+2x+1}{2x^2-3x+5}=\lim_{x\to\infty}\frac{4x^2}{2x^2}=\lim_{x\to\infty}2=2$$

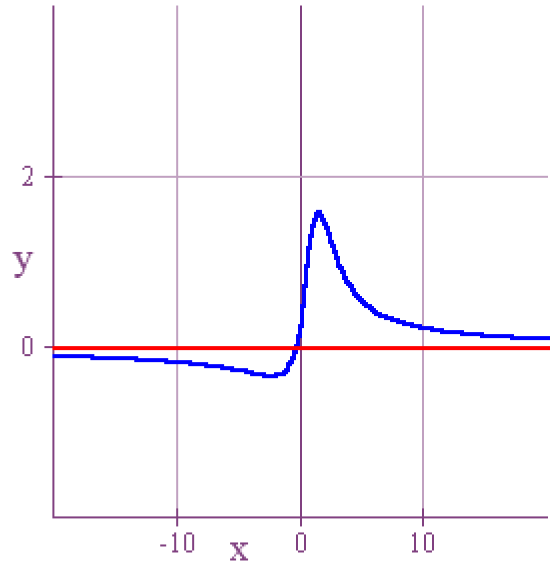
$$\lim_{x\to\infty}\frac{4x+2}{2x^2-3x+5}=\lim_{x\to\infty}\frac{4x}{2x^2}=\lim_{x\to\infty}\frac{2}{x}=0$$

$$\lim_{x\to\infty}\frac{4x^2+2x+1}{2x-3}=\lim_{x\to\infty}\frac{4x^2}{2x}=\lim_{x\to\infty}2x=\infty$$



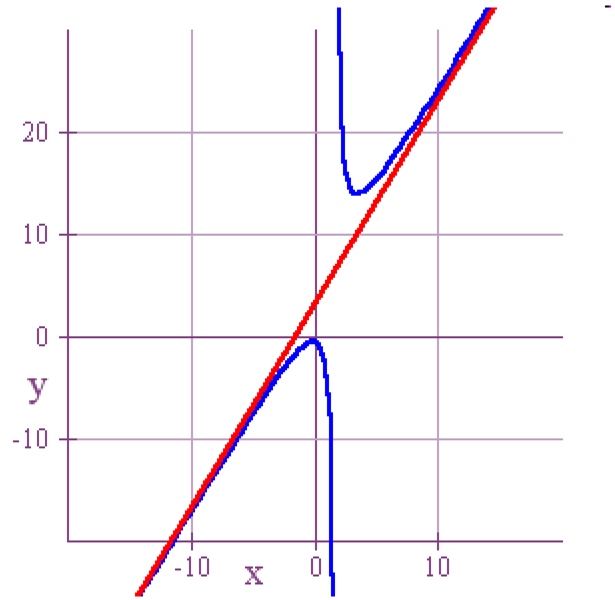
$$f(x) = \frac{4x^2 + 2x + 1}{2x^2 - 3x + 5}$$

$$\lim_{x \to \infty} \frac{4x^2 + 2x + 1}{2x^2 - 3x + 5} = \lim_{x \to \infty} \frac{4x^2}{2x^2} = \lim_{x \to \infty} 2 = 2$$



$$f(x) = \frac{4x+2}{2x^2-3x+5}$$

$$\lim_{x\to\infty} \frac{4x+2}{2x^2-3x+5} = \lim_{x\to\infty} \frac{4x}{2x^2} = \lim_{x\to\infty} \frac{2}{x} = 0$$



$$f(x) = \frac{4x^2 + 2x + 1}{2x - 3}$$

$$\lim_{x \to \infty} \frac{4x^2 + 2x + 1}{2x - 3} = \lim_{x \to \infty} \frac{4x^2}{2x} = \lim_{x \to \infty} 2x = \infty$$

It's So Simple!