DERIVATIVE FORMULAS 2

THEOREM (Chain Rule): If $y = f(g(x))$,

then $\dfrac{dy}{dx} = \dfrac{df}{dg} \cdot \dfrac{dg}{dx}$.

SIMPLIFIED PROOF: Suppose $y = y = f(g(x))$. Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

THEOREM (Chain Rule): If $y = f(g(x))$,

then $\dfrac{dy}{dx} = \dfrac{df}{dg} \cdot \dfrac{dg}{dx}$.

SIMPLIFIED PROOF: Suppose $y = y = f(g(x))$. Then

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x} = \frac{df}{dg} \cdot \frac{dg}{dx}.$$

Think of this as meaning that we multiply the derivative of the outer function by the derivative of the inner function.

EXAMPLES:

$$y = (3x + 2)^{10}$$

EXAMPLES:

$$y = (3x + 2)^{10}$$

$$\frac{dy}{dx} = 10(3x + 2)^{9} \cdot 3$$

EXAMPLES:

$$y = (4x^2 + 2x + 5)^{20}$$

EXAMPLES:

$$y = (4x^2 + 2x + 5)^{20}$$

$$\frac{dy}{dx} = 20(4x^2 + 2x + 5)^{19} \cdot (8x + 2)$$

EXAMPLES:

$$y = \sqrt{x^2 + 5x} = (x^2 + 5x)^{1/2}$$

EXAMPLES:

$$y = \sqrt{x^2 + 5x} = (x^2 + 5x)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}(x^2 + 5x)^{-1/2}(2x + 5) = \frac{2x + 5}{2\sqrt{x^2 + 5x}}$$

THEOREM: If $y = (fg)(x) = f(x) \cdot g(x)$,

then $\dfrac{dy}{dx} = f(x)\dfrac{dg}{dx} + g(x)\dfrac{df}{dx}$.

PROOF: Suppose $y = (fg)(x) = f(x) \cdot g(x)$. Then

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{(fg)(x+h)-(fg)(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(x+h)\cdot g(x+h) - f(x+h)\cdot g(x) + f(x+h)\cdot g(x) - f(x)\cdot g(x)}{h}$$

$$= \lim_{h\to 0} f(x+h)\frac{g(x+h)-g(x)}{h} + g(x)\frac{f(x+h)-f(x)}{h}$$

$$= f(x)\frac{dg}{dx} + g(x)\frac{df}{dx}.$$

THEOREM: If $y = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$,

then $\frac{dy}{dx} = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}$.

PROOF: Suppose $y = \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = f(x) \cdot g(x)^{-1}$. Then

$$\frac{dy}{dx} = f(x)\left[-g(x)^{-2} \cdot \frac{dg}{dx}\right] + g(x)^{-1} \cdot \frac{df}{dx}$$

$$= \frac{-f \cdot \frac{dg}{dx}}{g^2} + \frac{\frac{df}{dx}}{g} = \frac{-f \cdot \frac{dg}{dx}}{g^2} + \frac{g \cdot \frac{df}{dx}}{g^2} = \frac{g \cdot \frac{df}{dx} - f \cdot \frac{dg}{dx}}{g^2}.$$

SUMMARY 2

1. $$\frac{d(c)}{dx} = 0$$

2. $$\frac{dx^n}{dx} = nx^{n-1}$$

3. $$\frac{d(cf(x))}{dx} = c\frac{df}{dx}$$

4. $$\frac{d(f \pm g)}{dx} = \frac{df}{dx} \pm \frac{dg}{dx}$$

5. $$\frac{df(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

6. $$\frac{d(fg)}{dx} = f\frac{dg}{dx} + g\frac{df}{dx}$$

7. $$\frac{d\left(\frac{f}{g}\right)}{dx} = \frac{g\frac{df}{dx} - f\frac{dg}{dx}}{g^2}$$

EXAMPLES:

$$y = (4x^2 + 2x + 5)(x^3 + 1)$$

EXAMPLES:

$$y = (4x^2 + 2x + 5)(x^3 + 1)$$

$$\frac{dy}{dx} = (4x^2 + 2x + 5)(3x^2) + (x^3 + 1)(8x + 2)$$

EXAMPLES:

$$y = \frac{4x^2 + 2x + 5}{x^3 + 1}$$

EXAMPLES:

$$y = \frac{4x^2 + 2x + 5}{x^3 + 1}$$

$$\frac{dy}{dx} = \frac{(x^3 + 1)(8x + 2) - (4x^2 + 2x + 5)(3x^2)}{(x^3 + 1)^2}$$

$$= \frac{8x^4 + 2x^3 + 8x + 2 - 12x^4 - 6x^3 - 15x^2}{(x^3 + 1)^2}$$

$$= \frac{-4x^4 - 4x^3 - 15x^2 + 8x + 2}{(x^3 + 1)^2}$$

OR:

$$y = \frac{4x^2 + 2x + 5}{x^3 + 1} = (4x^2 + 2x + 5)(x^3 + 1)^{-1}$$

$$\frac{dy}{dx} = (4x^2 + 2x + 5)(-1)(x^3 + 1)^{-2}(3x^2) + (8x + 2)(x^3 + 1)^{-1}$$

$$= \frac{-12x^4 - 6x^3 - 15x^2}{(x^3 + 1)^2} + \frac{(8x + 2)(x^3 + 1)}{(x^3 + 1)^2}$$

$$= \frac{-12x^4 - 6x^3 - 15x^2}{(x^3 + 1)^2} + \frac{8x^4 + 2x^3 + 8x + 2}{(x^3 + 1)^2}$$

$$= \frac{-4x^4 - 4x^3 - 15x^2 + 8x + 2}{(x^3 + 1)^2}$$