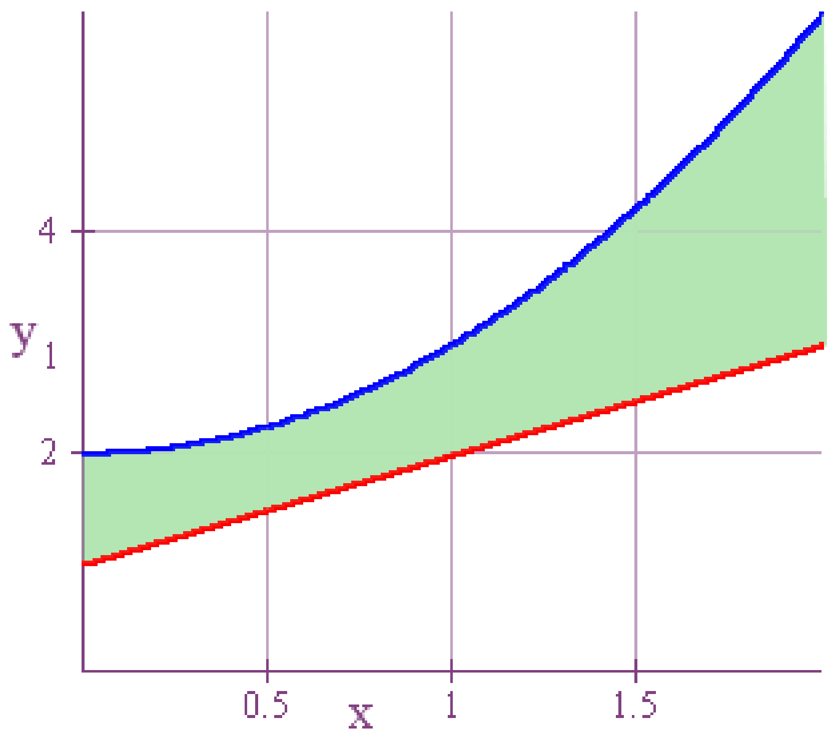
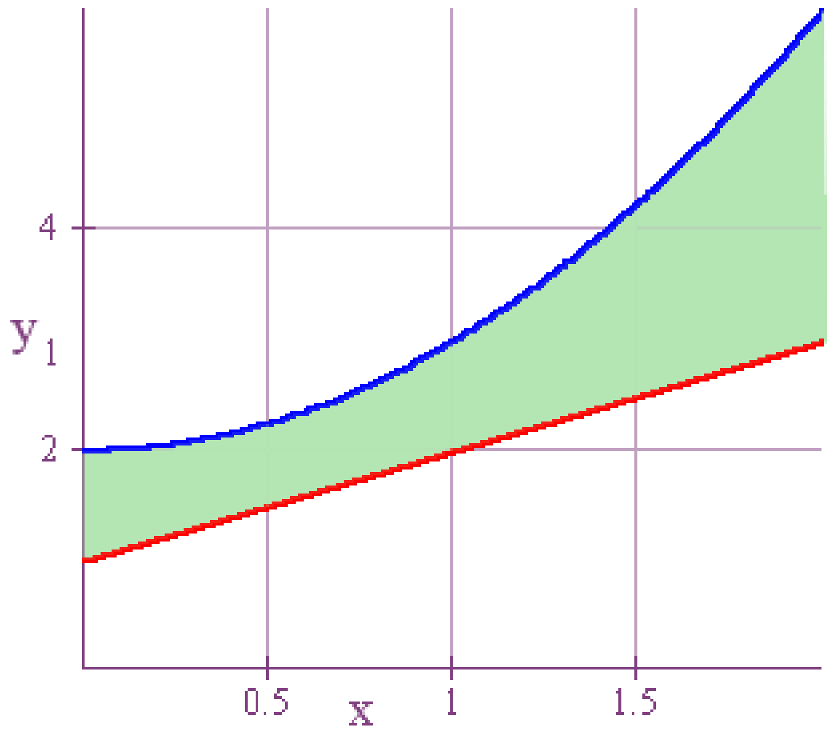
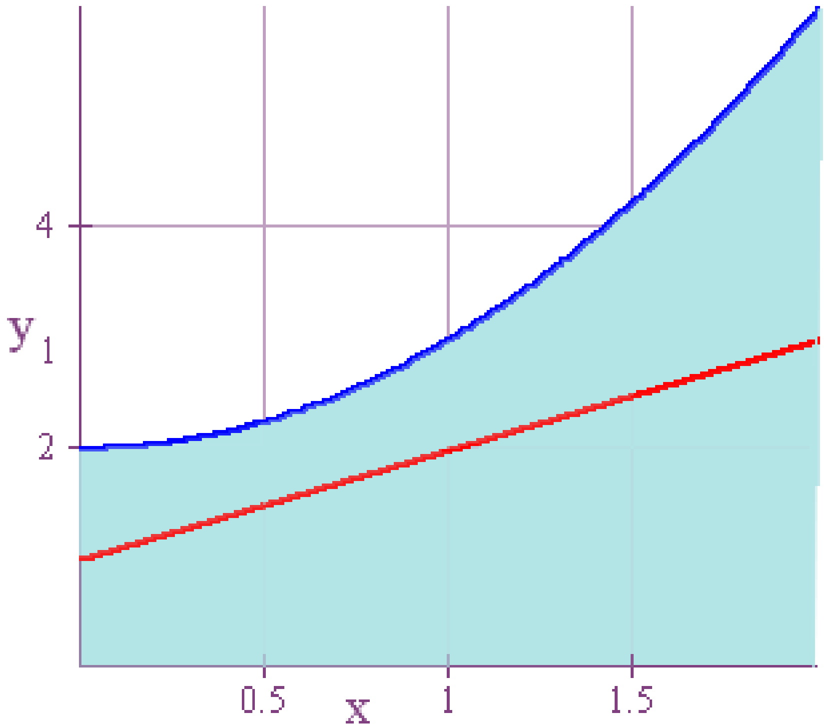
THE AREA BETWEEN TWO CURVES

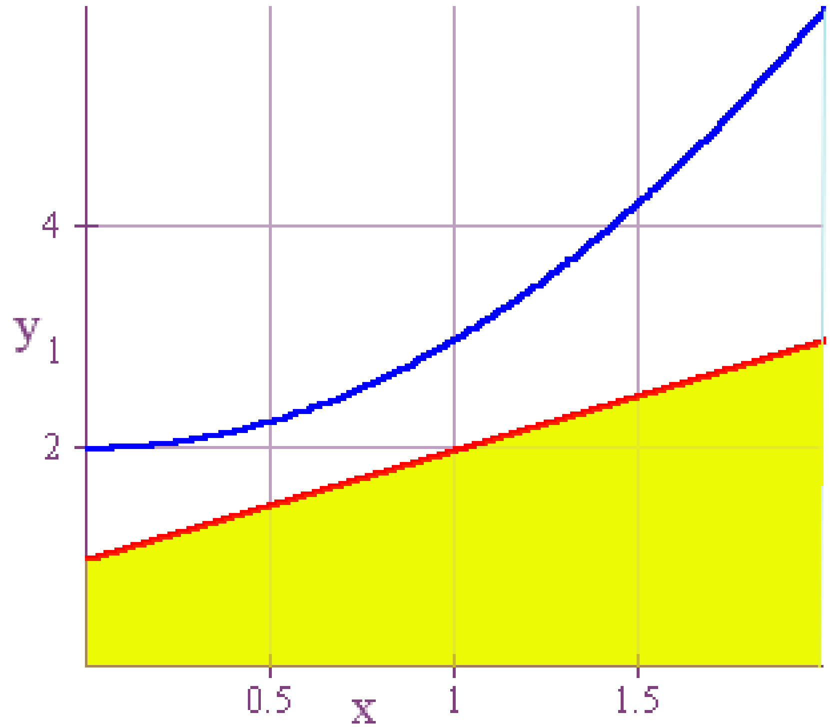


Suppose we want to find the area between the graphs of $y=x^2+2$ and $y=x+1$ on the interval from 0 to 2.



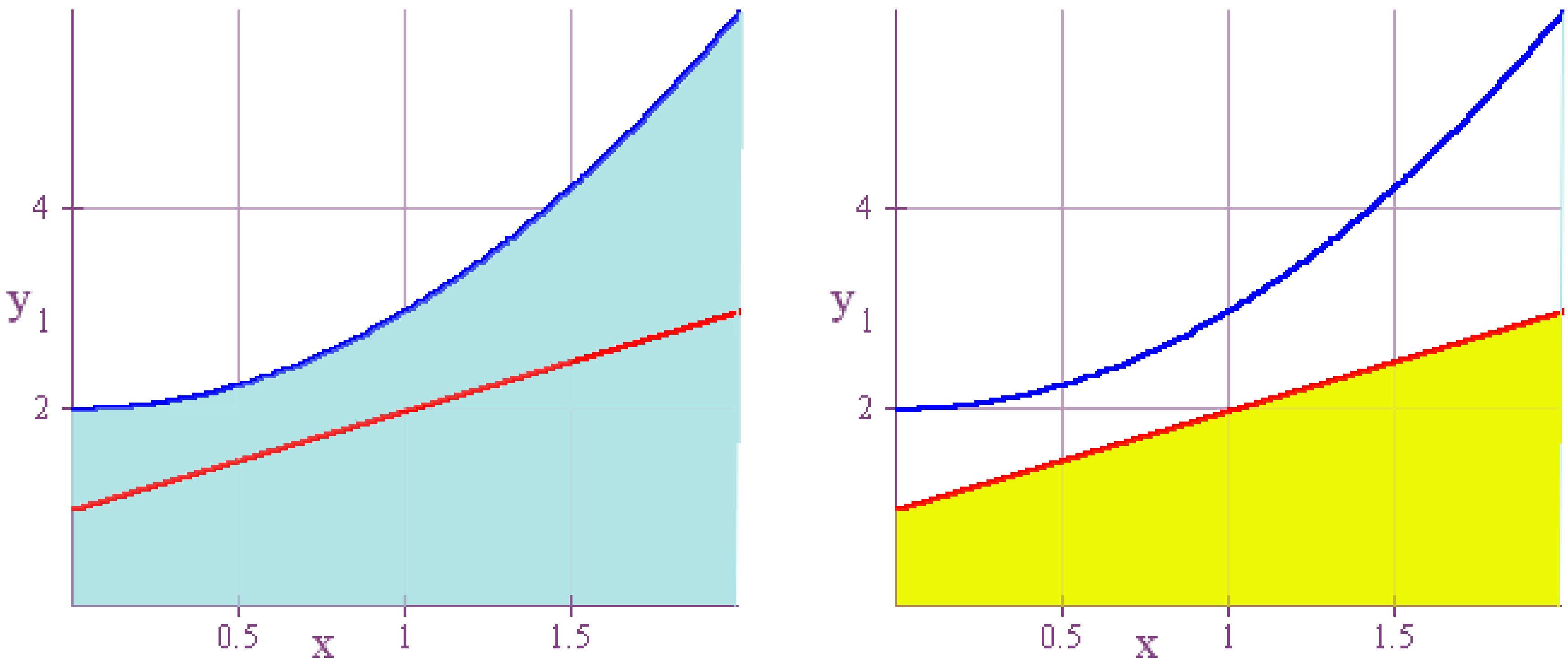
That’s easy to do. We just first find the area between $y=x^2+2$ and the x-axis, and then we subtract off the area between $y=x+1$ and the *x*=axis.





Also, we can express this area as the difference between two integrals.

$$\int_0^2 (x^2+2)\,dx - \int_0^2 (x+1)\,dx$$



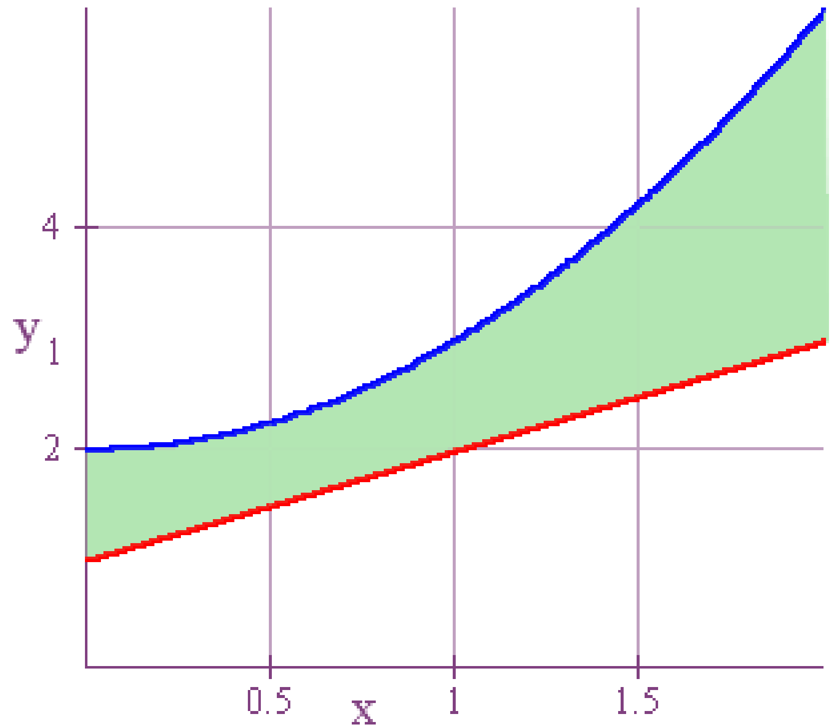
And finally, we can write this as a single integral that gives us the area between the two curves.

$$\text{Area} = \int_0^2 (x^2+2)\,dx - \int_0^2 (x+1)\,dx$$

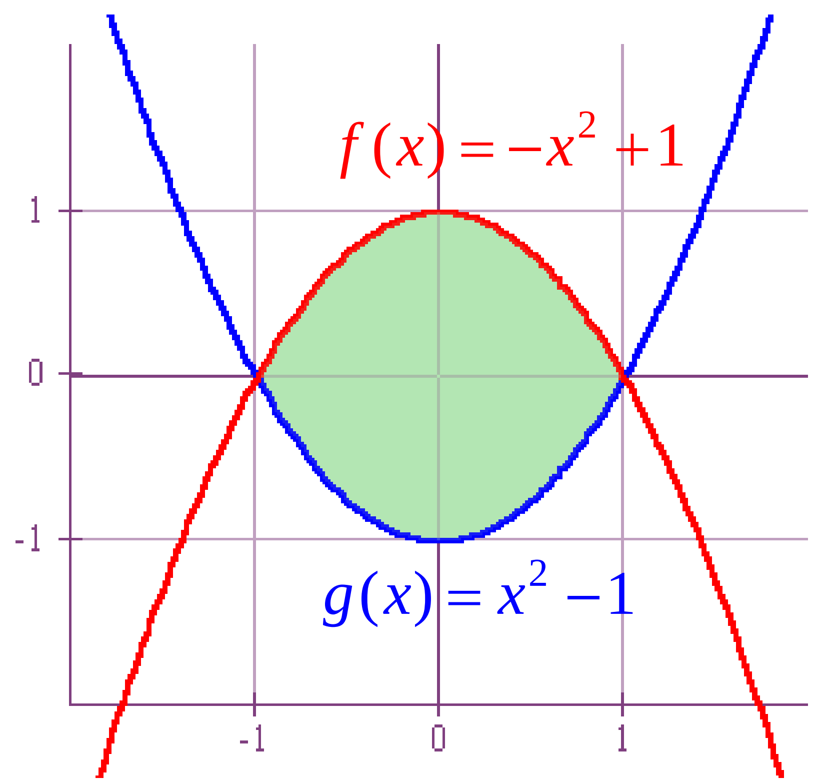
$$= \int_0^2 (x^2+2)-(x+1)\,dx$$

$$= \int_0^2 (x^2 - x + 1)\,dx$$

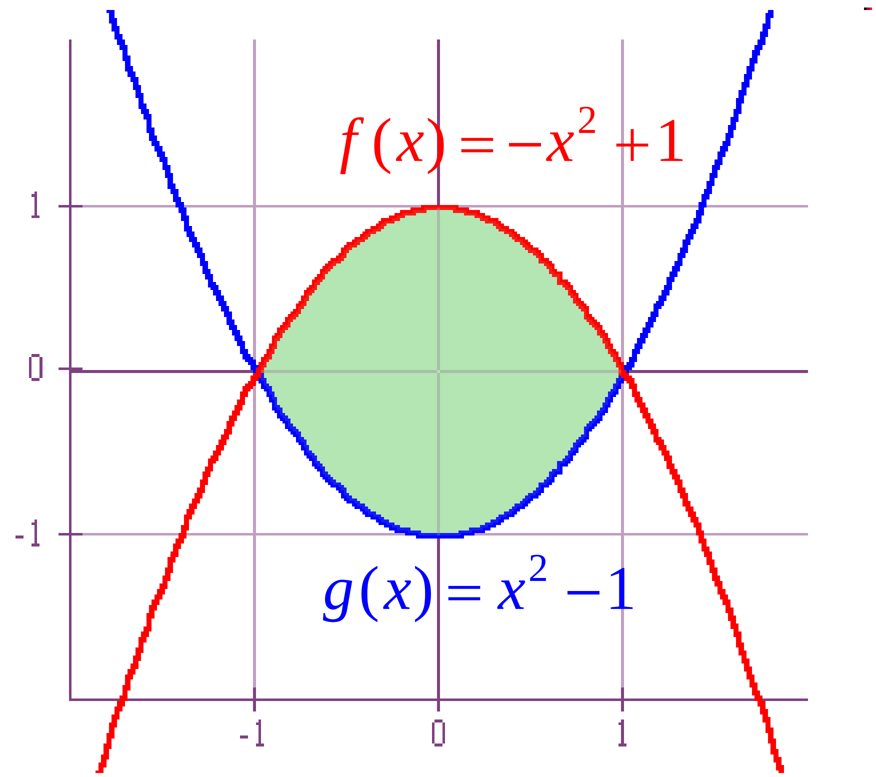
$$= \frac{x^3}{3} - \frac{x^2}{2} + x \Big|_0^2 = \frac{8}{3}$$



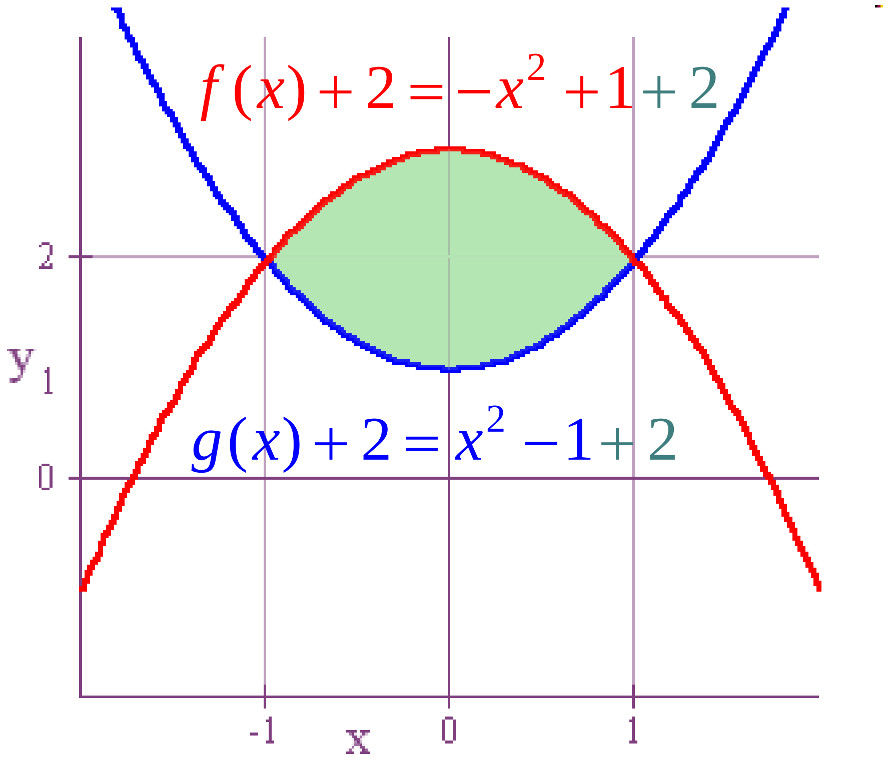
Now let's suppose that the region we want to find the area of is not entirely above the *x*-axis.



If our region is not entirely above the *x*-axis, then we can't interpret the integral as representing area as we did before.

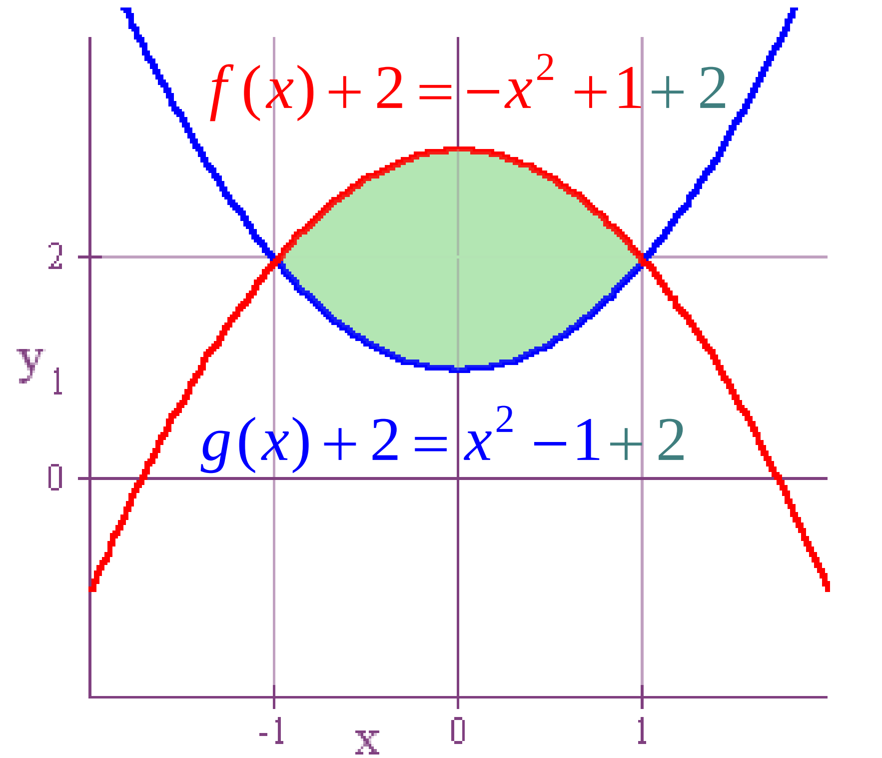


However, by adding an appropriate constant to both functions, we can create a vertical shift that places the region entirely above the *x*-axis.



And now we can proceed as before. We can find the area between the two curves by subtracting the bottom function from the top.

$$\text{Area} = \int_{-1}^{1} (f(x)+2) - (g(x)+2)\,dx$$



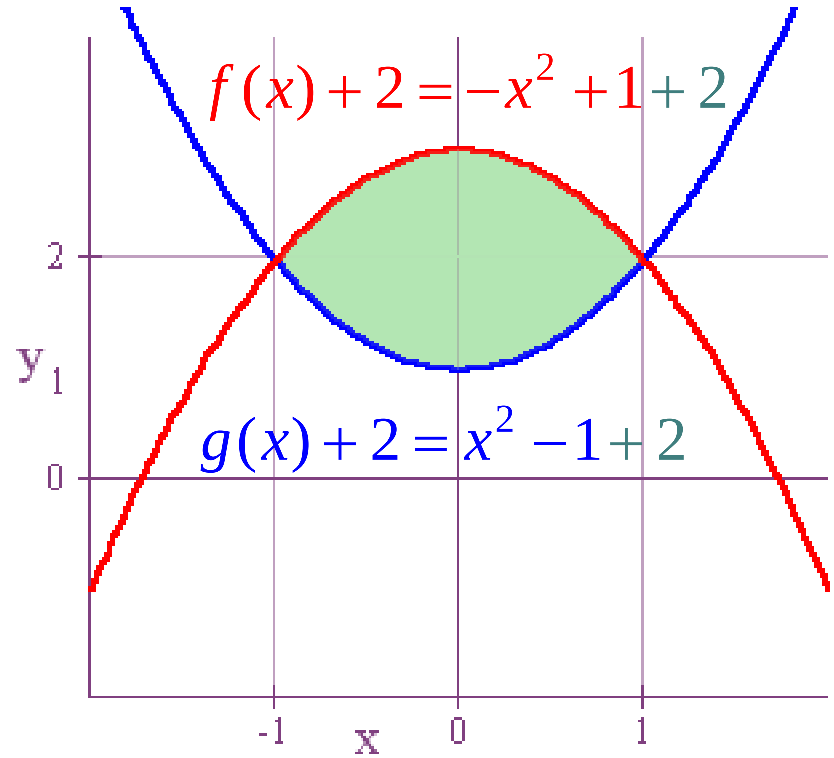
However, when we do this the constant we added on simply subtracts off.

$$\text{Area} = \int_{-1}^{1} (f(x)+2)-(g(x)+2)\,dx$$

$$= \int_{-1}^{1} (f(x)-g(x))\,dx$$

$$= \int_{-1}^{1} (-x^2+1)-(x^2-1)\,dx$$

$$= \int_{-1}^{1} (-2x^2+2)\,dx = \frac{8}{3}$$



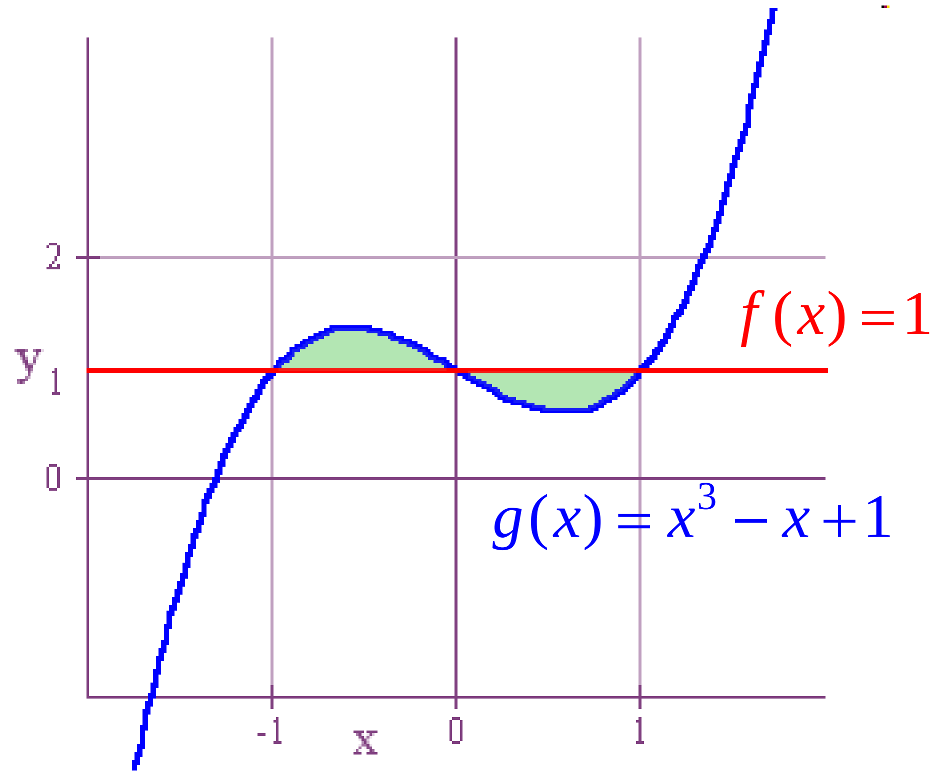
The bottom line is that we don't need to worry about whether the region is below the *x*-axis or not. To find the area we just subtract the bottom function from the top.

$$\text{Area between curves } = \int_a^b (f(x) - g(x))\,dx$$

top function

bottom function

Now let's look at this example. The problem is that g(x) is the top function on the interval from -1 to 0, but it's the bottom function on the interval from 0 to 1.



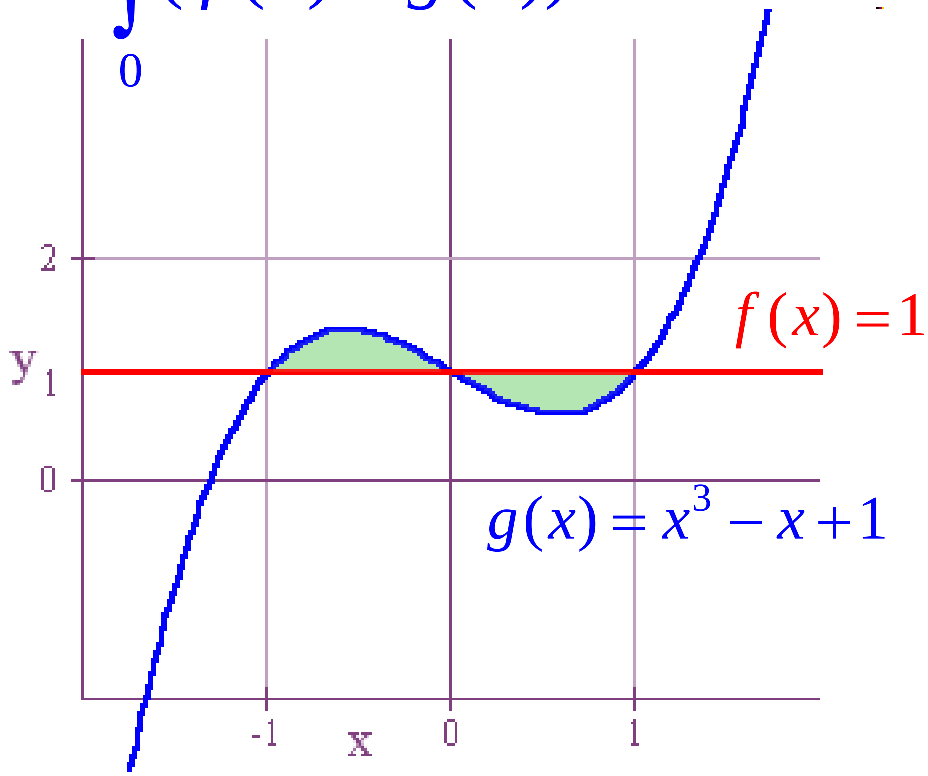
So what do we do? Simple! We just do two separate integrals.

$$\text{Area} = \int_{-1}^{0} (g(x) - f(x))\,dx + \int_{0}^{1} (f(x) - g(x))\,dx$$

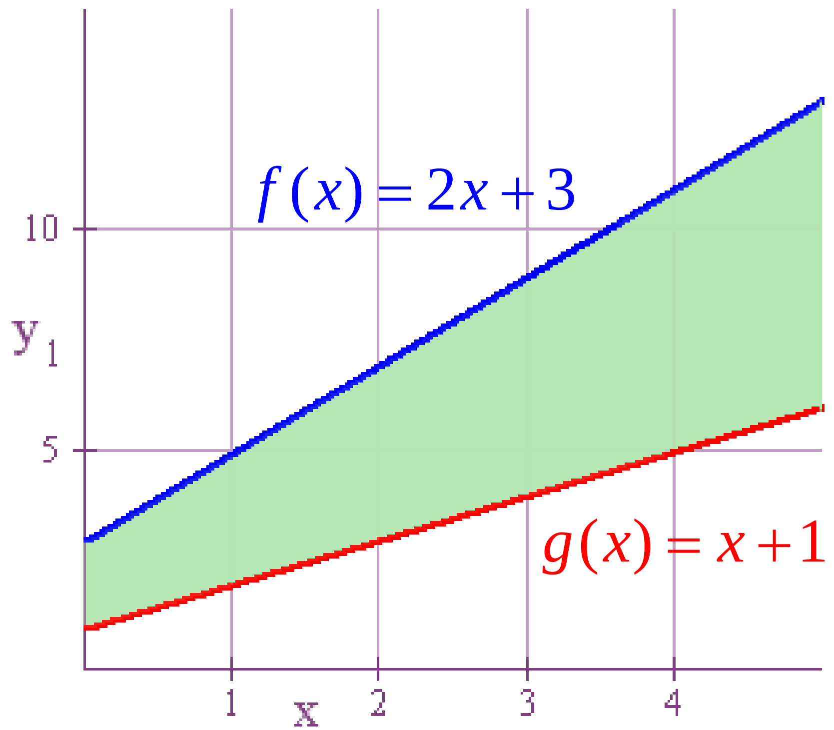
$$= \int_{-1}^{0} ((x^3 - x + 1) - 1)\,dx$$

$$+ \int_{0}^{1} (1 - (x^3 - x + 1))\,dx$$

$$= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$



Now let's suppose that you have a business that over a 5 year period generates revenue at a rate of f(x)=2x+3 dollars per year and costs at a rate of g(x)=x+1 dollars per year. What is your profit over this five year period?



$$\text{Revenue} = \int_0^5 (2x+3)\,dx, \quad \text{Cost} = \int_0^5 (x+1)\,dx$$

$$\text{Profit} = \text{Revenue - Cost} = \int_0^5 (2x+3)\,dx - \int_0^5 (x+1)\,dx$$

$$= \int_0^5 (2x+3)-(x+1)\,dx$$

$$= \int_0^5 (x+2)\,dx = \left.\frac{x^2}{2}+2x\right|_0^5$$

$$= \$22.50$$