PRACTICE – NORMAL SUBGROUPS AND HOMOMORPHISMS – ANSWERS

1. Prove: If $f: G \to H$ is a homomorphism and $a \in G$, then $f(a^{-1}) = f(a)^{-1}$.

   Proof: Let G and H be groups, $a \in G$, and let $f: G \to H$ be a homomorphism. Then $e = f(a) \cdot f(a)^{-1}$ and $e = f(e) = f(aa^{-1}) = f(a)f(a^{-1})$. Therefore, since elements in groups have unique inverses, it follows that $f(a^{-1}) = f(a)^{-1}$. □

2. Prove: Let G_1 and G_2 be groups (not necessarily finite), let $f: G_1 \to G_2$ be a homomorphism from G_1 onto G_2, and let $K = \{x \in G_1 \mid f(x) = e_2\}$. Then K is a subgroup of G_1.

   Proof: In order to show that K is a subgroup of G, we need to establish both closure and the existence of inverses. Thus, suppose $x, y \in K$. Then $f(xy) = f(x)f(y) = e_2e_2 = e_2$, the identity element in G_2. Therefore, $xy \in K$ and K is closed under multiplication. Now suppose $x \in K$ and consider $x^{-1} \in G_1$. Clearly, $e_2 = f(e_1) = f(xx^{-1}) = f(x)f(x^{-1}) = e_1 \cdot f(x^{-1}) = f(x^{-1})$. Therefore, $x^{-1} \in K$, and K is a subgroup of G. □