PRACTICE – CONJUGATES AND COMMUTATORS – ANSWERS

1. If $a=(1\quad 7)$ and $b=(1\quad 2\quad 3)$, find the conjugate aba^{-1}

   $(1\quad 7)(1\quad 2\quad 3)(1\quad 7)=(2\quad 3\quad 7)$

2. If $a=(1\quad 7)$ and $b=(1\quad 2\quad 3)$, find the commutator $aba^{-1}b^{-1}$.

   $(1\quad 7)(1\quad 2\quad 3)(1\quad 7)(3\quad 2\quad 1)=(2\quad 3\quad 7)(3\quad 2\quad 1)=(1\quad 3\quad 7)$

3. <u>Prove:</u> If *G* is a group (not necessarily finite), *H* is a subgroup of *G*, and $a\in G$, then aHa^{-1} is also a subgroup of *G*.

   <u>Proof:</u> In order to show that aHa^{-1} is a subgroup of *G*, we need to establish both closure and the existence of inverses. Thus, suppose that $b,c\in aHa^{-1}$. Then there exist *x* and *y* in *H* such that $b=axa^{-1}$ and $c=aya^{-1}$. Hence, $bc=(axa^{-1})(aya^{-1})=a(xy)a^{-1}\in aHa^{-1}$ since $xy\in H$. Now suppose that $x\in H$. Then it is also the case that $x^{-1}\in H$. Hence, axa^{-1} and $ax^{-1}a^{-1}$ are both elements of aHa^{-1}. Furthermore, $(axa^{-1})(ax^{-1}a^{-1})=ax(aa^{-1})x^{-1}a^{-1}=a(xx^{-1})a^{-1}=aa^{-1}=e$. Therefore, inverses exist in aHa^{-1}, and aHa^{-1} is a subgroup of *G*. □

4. <u>Prove:</u> If *G* is a group (not necessarily finite), $a\in G$, and $x\in G$ such that the order of $\langle x\rangle$ in *G* is $|\langle x\rangle|=n$, then the order of the conjugate of *x*, axa^{-1}, is also *n*.

   <u>Proof:</u> Let *G* be a group, $a\in G$, and $x\in G$ such that the order of $\langle x\rangle$ in *G* is $|\langle x\rangle|=n$, and consider the order of axa^{-1}. If we raise this element to n^{th} power, then we obtain $(axa^{-1})^n=axa^{-1}\cdot axa^{-1}\cdot axa^{-1}\cdot\dots\cdot axa^{-1}=ax^na^{-1}=aa^{-1}=e$. Additionally, if there were a natural number $m<n$ such that $(axa^{-1})^m=e$, then this would imply that $x^m=[a^{-1}(axa^{-1})a]^m=a^{-1}(axa^{-1})^m a=a^{-1}a=e$ which contradicts our assumption that $|\langle x\rangle|=n$. Therefore, the order of the conjugate of *x*, axa^{-1}, is also *n*. □

5. <u>Prove:</u> If *G* is a group (not necessarily finite), and $a,b\in G$, then $aba^{-1}=e$ if and only if $b=e$.

   <u>Proof:</u> Suppose *G* is a group and $a,b\in G$. Since our claim involves “if and only if,”

we'll have to prove the implication in both directions.

$(\Rightarrow)$ Suppose $aba^{-1}=e$. Then $a^{-1}\left(aba^{-1}\right)a=a^{-1}ea \Rightarrow ebe=a^{-1}a \Rightarrow b=e$.

$(\Leftarrow)$ Suppose $b=e$. Then $aba^{-1}=aea^{-1}=aa^{-1}=e$.

Therefore, $aba^{-1}=e$ if and only if $b=e$. $\square$